SIO203C: PDE Notes A

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Lecture 3

Quasilinear equations

We are now going to develop the method of characteristics for quasilinear equations

$$a(x,y,z)z_x + b(x,y,z)z_y = c(x,y,z)\,. \tag{3.1}$$

in detail. Our earlier "semilinear" examples are special cases of (3.1).

We follow Lagrange and view the problem geometrically. The solution, $z = z(x,y)$, of (3.1) is a surface in the three-dimensional space (x,y,z). The normal to this *solution surface* is the vector

$$\boldsymbol{n} = (-z_x, -z_y, 1)\,. \tag{3.2}$$

The quasilinear equation (3.1) is equivalent to

$$\boldsymbol{a}\cdot\boldsymbol{n} = 0\,, \tag{3.3}$$

where $\boldsymbol{a} \equiv (a,b,c)$ is a vector field associated with (3.1). In other words, $\boldsymbol{a}$, is normal to $\boldsymbol{n}$ and hence $\boldsymbol{a}$ lies in the plane tangent to the solution surface.

In this lecture we denote the independent variable by z to emphasize that the solution surface of (3.1) is in the three-dimensional space (x,y,z).

The characteristic curves (CCs) are everywhere tangent to vector $\boldsymbol{a}$. In fluid mechanics $\boldsymbol{a}$ is a velocity field and a CC is a streamline. Notice that CCs live in the space (x,y,z). The projections of these three-dimensional CCs down onto the (x,y)-plane are the *characteristic base*

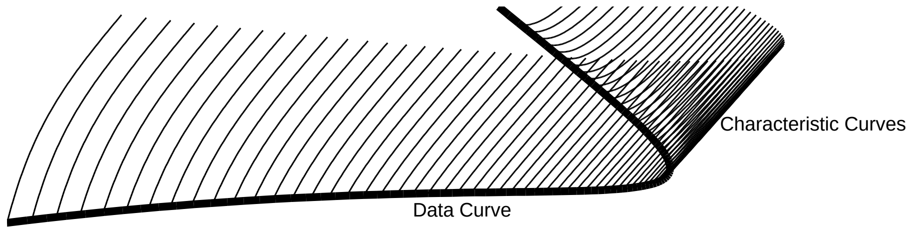

Figure 3.1: The totality of characteristics passing through the data curve is the solution surface. `characteristics.eps`

curves (CBCs). Sometimes, confusingly, people also refer to the characteristic base curves as "characteristics" — I did this myself in the first lecture. I'll sincerely try to maintain the distinction between CCs and CBCs in this lecture.

Yikes! ccs, the dc, cbcs & the dbc

The other important curve associated with the PDE (3.1) is the *data curve* (DC). The DC is determined by the initial or boundary conditions and also lives in the three-dimensional space (x, y, z). For example, if one requires the solution of (3.1) with the initial condition that $z(0, y) = 2y$ then the data curve is a straight line determined by the intersection of the two planes $x = 0$ and $z = 2y$. Just as the characteristic base curves lie in the (x, y) plane, there is also a *data base curve* (DBC). Since it is much easier to sketch in two-dimensions than in three, we frequently visualize PDE problems by sketching the CBCs and the DBC.

I have attempted to illustrate Lagrange's geometric vision for solving PDE's in figure 3.1. In the streamline analogy, you can imagine that the DC is source of dye which colours each passing streamline. The painted surface is the solution surface of (3.1).

3.1 Characteristics and their properties

On a CC the PDE reduces to an ODE. Points on the CCs are $[x(t), y(t), z(t)]$ where x, y and z satisfy Lagrange's equations

Lagrange's equations

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}t} = a\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = b\,, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = c\,.} \tag{3.4}$$

The equations above ensure that the vector field $\boldsymbol{a} = (a, b, c)$ is everywhere tangent to a CC.

Every surface consisting wholly of CCs is a solution surface. Conversely, every solution surface, $z = z(x, y)$, is also made wholly of CCs. To see this, suppose we that we move around on the solution surface by solving

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a[x, y, z(x, y)], \quad \frac{\mathrm{d}y}{\mathrm{d}t} = b[x, y, z(x, y)]. \tag{3.5}$$

Computing $\mathrm{d}z/\mathrm{d}t$ we have

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &= z_x \frac{\mathrm{d}x}{\mathrm{d}t} + z_y \frac{\mathrm{d}y}{\mathrm{d}t}, && \text{(chain rule)} \\ &= a z_x + b z_y, && \text{from (3.5)}, \\ &= c, && \text{from (3.1).} \end{aligned} \tag{3.6}$$

Thus we recover Lagrange's equations, (3.4).

The CCs are a two parameter family of curves. One parameter is t in (3.4) and the second parameter, s, labels the initial condition: $x = x(s, t)$, $y = y(s, t)$ and $z = z(s, t)$. Elimination of s and t produces the solution surface $z = z(x, y)$. Thus the geometric vision is that the solution surface is generated by a family of CCs parametrized by s.

A quasilinear recipe

We automate the solution of the quasilinear PDE (3.1) with the recipe:

- parameterize the DC: $x = x(s, 0)$, $y = y(s, 0)$ and $z = z(s, 0)$;
- solve Lagrange's equations (3.4) using the initial condition from above;
- eliminate the parameters s and t and get $z = z(x, y)$;
- check by substitution.

This recipe is a formulaic version of the total-derivative procedure used in the first lecture to reduce PDEs to ODEs. The parameter t we introduce in Lagrange's equations is just a way of treating x and y evenhandedly.

Let's use the recipe in an example.

Example: Solve the quasilinear PDE:

$$z_x + z_y = z\,, \quad z(0, y) = y^2\,. \tag{3.7}$$

We parameterize the DC by writing

$$t = 0: \quad x = 0\,, \quad y = s\,, \quad z = s^2\,. \tag{3.8}$$

As we vary s we move along the DC. Next, we solve Lagrange's equations with the initial condition in (3.8)

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 1\,, \quad \Rightarrow \quad x = t\,; \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 1\,, \quad \Rightarrow \quad y = t + s\,; \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= z\,, \quad \Rightarrow \quad z = s^2 \mathrm{e}^t\,. \end{aligned} \tag{3.9}$$

The third step is to eliminate the parameters s and t:

$$t = x\,, \quad s = y - x\,. \tag{3.10}$$

Eliminating s and t in the third equation in (3.7) gives $u = (y-x)^2 \exp(x)$. The final step is to check by substitution. ■

Example: In lecture 1 our approach was somewhat clumsier than the quasilinear recipe. Let's illustrate the differences by discussing the particular example

$$(x^2 + y^2) z_x - 2xy z_y = 0\,. \tag{3.11}$$

If we divide by $x^2 + y^2$ then the left hand side can be regarded as a total derivative:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = z_x + \frac{\mathrm{d}y}{\mathrm{d}x} z_y = 0\,, \tag{3.12}$$

where

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy}{x^2 + y^2}\,. \tag{3.13}$$

To find the CCs we have to integrate the ODE above. We can do this using tricks from chapter 1 of **BO** — the ODE is scale invariant i.e., unchanged by $x \to ax$ and $y \to ay$.

In (3.13) are treating x and y differently i.e., we're regarding y as a function of x on the CC, rather than the reverse. This might lead to issues if the CCs bend around so that there are several values of y for one value of x. And the ODE in (3.13) has a singularity at the origin which is not in the original PDE.

The alternative approach is to look for a parameterized solution for the CCs. This is the second step of the quasilinear recipe. This alternative route presents us with Lagrange's equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 + y^2 , \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -2xy , \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = 0 . \tag{3.14}$$

Of course, if we divide $\mathrm{d}y/\mathrm{d}t$ by $\mathrm{d}x/\mathrm{d}t$ then we return to (3.13). But another way to solve (3.14), and thus determine the CCs, is to use the new variables $a = x + y$ and $b = x - y$:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = b^2 , \qquad \frac{\mathrm{d}b}{\mathrm{d}t} = a^2 , \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(a^3 - b^3\right) = 0 . \tag{3.15}$$

Hence $y^3 + 3x^2y$ is constant on characteristics, and the solution is that

$$z = \text{an arbitrary function of } y^3 + 3x^2y \blacksquare \tag{3.16}$$

Example: Solve

$$(x^2 + y^2)z_x - 2xyz_y = 0 . \tag{3.17}$$

with $z(x, x) = \exp(x)$. The general solution is $z(x, y) = a(y^3 + 3x^2y)$ and the arbitrary function a is determined by applying the data:

$$\mathrm{e}^x = a\left(4x^3\right) , \quad \Rightarrow \quad a(z) = \mathrm{e}^{(z/4)^{1/3}} ,$$

where the one-third power above is defined as $z^{1/3} = \mathrm{sgn}(z)|z|^{1/3}$.

More dimensions

The generalization to dimension d is clear. The quasilinear PDE is

$$\sum_{n=1}^{d-1} a_n(\boldsymbol{x})\frac{\partial z}{\partial x_n} = f(\boldsymbol{x}) , \tag{3.18}$$

where a point in the d-dimensional solution space is

$$\boldsymbol{x} = (x_1, x_2, \cdots, x_{d-1}, z) . \tag{3.19}$$

Lagrange's equations are

$$\frac{\mathrm{d}x_n}{\mathrm{d}t} = a_n , \quad (n = 1, \cdots d-1) , \tag{3.20}$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = f . \tag{3.21}$$

As initial conditions, we have to supply data on a $(d-2)$-dimensional surface, which can be parameterized with s_1, $s_2 \cdots s_{d-2}$. The general solution will be an arbitrary function with $d-2$ variables.

Example: Find the general solution of the PDE:

$$xyu_x + zu_y + u_z = 0. \tag{3.22}$$

In this example the solution surface lives in the four-dimensional space (x, y, z, u). Lagrange's equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = xy, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = z, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = 1, \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = 0. \tag{3.23}$$

In this problem $d = 4$, and we seek a general solution in the form

$$z = \text{an arbitrary function of two variables.} \tag{3.24}$$

The two variables above correspond to the two-dimensional data surface, and this surface is carried into the four-dimensional space (x, y, z, u) by the CCs. Thus the solution "surface" is three dimensional. Because u is constant on a CC, the two arbitrary functions above are also constant on a CC.

You might be tempted to begin your assault on Lagrange's equations by dividing the first two equations above to obtain

$$\frac{\mathrm{d}x}{x} = \frac{y\mathrm{d}y}{z}. \tag{3.25}$$

But don't treat z as a constant! This is a dumb mistake because x, y, z and u all vary with t along a characteristic. Thus one cannot integrate the equation above.

Instead notice that Lagrange's equations tell us that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(y - \tfrac{1}{2}z^2\right) = 0, \tag{3.26}$$

so that

$$y - \tfrac{1}{2}z^2 = A, \tag{3.27}$$

where A is constant on a CC. Thus

$$\frac{\mathrm{d}x}{\mathrm{d}z} = xy = x\left(A + \tfrac{1}{2}z^2\right), \tag{3.28}$$

or

$$\ln x = B + Az + \tfrac{1}{6}z^3. \tag{3.29}$$

The constants of integration A and B are both constant along CCs. We can eliminate A between (3.27) and (3.29) and obtain

$$\ln x = B + yz - \tfrac{1}{3}z^3. \tag{3.30}$$

At this point (3.27) and (3.30) define two function $A(x, y, z)$ and $B(x, y, z)$ which are both solutions of the PDE. Thus the general solution of the PDE is

$$u = a\left(y - \tfrac{1}{2}z^2, \ln x - yz + \tfrac{1}{3}z^3\right), \tag{3.31}$$

where a is arbitrary ■

Example: Find the solution of the PDE:

$$xyu_x + zu_y + u_z = z\,, \tag{3.32}$$

with the data $u(x,y,0) = 0$.

3.2 More theory

Does the quasilinear recipe always work? No, because sometimes the problem has no solution and other times there is an infinitude of solutions. Let's recall a simple example from lecture 1: find the function $z(x,y)$ satisfying the PDE:

$$z_x = 0, \qquad z(x,0) = x\,. \tag{3.33}$$

It is obvious that this equation doesn't have a solution: differentiating the the data with respect to x, we discover that $z_x = 1$. Thus the DC contradicts the equation.

In (3.33) the DBC coincides with a CBC. But the DC is not a CC: only the base curves coincide and the problem is inconsistently posed. To lift the solution away from the DC the vector field $\boldsymbol{a}$ in (3.3) must pass through DC.

If you apply the quasilinear recipe to the simple example above then the first place you get into trouble is in trying to solve for x and y in terms of t and s. It can happen that the mapping from the (s,t) plane to the (x,y) plane is not invertible. The condition for invertibility is that

$$J \equiv \frac{\partial(x,y)}{\partial(s,t)} = x_s y_t - x_t y_s \neq 0\,. \tag{3.34}$$

J is the *Jacobian* of the mapping $(s,t) \to (x,y)$. But the Lagrange's equations tell us that

$$J = bx_s - ay_s\,. \tag{3.35}$$

Hence, in order to use the recipe, we must have $bx_s - ay_s \neq 0$ on the DC.

The condition $J \neq 0$ is equivalent to requiring that the data base curve is nowhere tangent to a characteristic base curve. In the example surrounding (3.33) this condition is strongly violated because the data base curve is a characteristic base curves: the two curves are tangent everywhere! More subtle problems occur if there are isolated points on the data base curve at which $J = 0$ i.e., local points of tangency. This pathology is exhibited by the PDE in problem 3.6: $J = 0$ at the isolated point $(x, y) = (0, 0)$ on the DBC.

It all starts to seem clear: if $J \neq 0$ on DC then we can start on the DC and confidently step the into the wild blue yonder. By this I mean that if $J \neq 0$ on DC then the PDE has a solution in the neighbourhood of the DC. In other words, the solution exists *locally*. But if $J = 0$ somewhere on the DBC then we anticipate problems. For instance, the PDE in problem 3.6 has no solution at some points which are arbitrarily close to the DBC (see right panel of figure 3.5). And in the example (3.33) there is no solution at all. But, just to show that there are still surprises, we turn to yet another example.

Example Consider the PDE:

$$3(z - y)^2 z_x - z_y = 0\,, \quad z(0, y) = y\,. \tag{3.36}$$

We parameterize the DC with

$$t = 0: \quad x = 0\,, \quad y = s\,, \quad z = s\,. \tag{3.37}$$

Before pressing on with the recipe, we calculate J on the DC

$$J = bx_s - ay_s = (-1) \times 0 - \underbrace{3(z - y)^2}_{\text{zero on the DC}} \times 1 = 0\,. \tag{3.38}$$

We should be in trouble: this is as bad as it gets because J is zero everywhere on the DC. But blithely ignoring the red light in (3.38) and continuing with the recipe one very quickly finds that

$$z = y + x^{1/3} \tag{3.39}$$

is a solution of (3.36). In other words the recipe works, even though $J = 0$ everywhere on the DC! This example shows that $J = 0$ is necessary, but not sufficient, condition for nonexistence of a solution in the neighbourhood of the DC ■

The point of the example above is that our conclusion following (3.35) assumes that $z(x, y)$ has continuous first derivatives on the DC, and the solution in (3.39) does not. In this example the CBCs are tangent to the DBC everywhere. But still the CC's manage to leave the DC and make a solution surface. The DBC is a *caustic* or a *envelope* of the CBC. Thus even if $J = 0$ on the DC, then we can still solve the PDE provided the DBC is a caustic of CBCs.

3.3 The nonlinear advection equation

We turn now to a detailed study of an important example quasilinear equation. This is the nonlinear advection equation

$$\boxed{u_t + uu_x = 0 \qquad \text{or} \qquad u_t + \left(\tfrac{1}{2}u^2\right)_x = 0\,.} \tag{3.40}$$

The final expression in (3.40) shows that the nonlinear advection equation can be written as a conservation equation for the density u, and the flux $u^2/2$.

Comparing the linear advection equation

$$u_t + cu_x = 0 \tag{3.41}$$

with nonlinear advection equation (3.40) we see that in the nonlinear case the speed of the disturbance is $c = u$. Thus, loosely speaking, we anticipate that in the nonlinear case bigger disturbances will travel more rapidly.

We use the method of characteristics to solve (3.40) with the initial condition that

$$u(x,0) = F(x)\,, \tag{3.42}$$

where $-\infty < x < \infty$. My discussion follows Whitham, though I am solely responsible for the myrmecophilia.

Imagine an ant moving along some curve $x = x(t)$ in the (x,t)-plane. The solution of (3.40), $u(x,t)$, can be visualized as a surface lying above that plane. The value of $u(x,t)$ observed by the moving ant is obtained from the "total derivative" of $u(x,t)$, namely

AKA the "advective derivative"

$$\frac{\mathrm{d}}{\mathrm{d}t}u\left(x(t),t\right) = u_t\left(x(t),t\right) + \frac{\mathrm{d}x}{\mathrm{d}t}u_x\left(x(t),t\right)\,. \tag{3.43}$$

We now consider a mathematically inclined ant, starting at $x(0) = \xi$, who adjusts her trajectory so that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u(x,t)\,, \quad x(0) = \xi\,. \tag{3.44}$$

Because $u(x,t)$ satisfies the advection equation (3.40) this ant observes that

$$\frac{\mathrm{d}}{\mathrm{d}t}u\left(x(t),t\right) = 0\,. \tag{3.45}$$

On the characteristics (3.44) pde (3.40) reduces to ode (3.45).

In other words, on a curve determined by (3.44), $u(x,t)$ is constant. In fact, since u satisfies the initial condition (3.42), the constant value of u is just

$$u(x,t) = F(\xi)\,. \tag{3.46}$$

In other words, the initial value of u is $F(\xi)$ and if ant picks her trajectory according to (3.44) then u doesn't change.

Now that we realize u is constant on the trajectory, it is trivial to determine that trajectory by integrating (3.44):

$$x = \xi + ut\,. \tag{3.47}$$

Finally, eliminate $\xi = x - ut$ between (3.46) and (3.47) to get u as a function of x and t. This gives

$$u = F(x - ut)\,. \tag{3.48}$$

Given F we can, in principle, solve the equation above for $u(x,t)$ (examples follow). After sorting out some notational distractions, you can also obtain this solution using the quasilinear solution earlier in this lecture.

It is instructive to check by substitution that (3.48) solves the PDE (3.40). Taking an x-derivative of (3.48) we get:

$$u_x = (1 - u_x t)F'(x - ut)\,, \quad \Rightarrow \quad u_x = \frac{F'(\xi)}{1 + tF'(\xi)}\,. \tag{3.49}$$

And the t-derivative of (3.48) is

$$u_t = -(tu)_t F'(x - ut)\,, \quad \Rightarrow \quad u_t = -\frac{uF'(\xi)}{1 + tF'(\xi)} = -uu_x\,. \tag{3.50}$$

A kinky initial condition

Suppose the initial condition is

$$u = \begin{cases} 1 - |x|\,, & \text{if } |x| < 1, \\ 0\,, & \text{if } |x| > 1,. \end{cases} \tag{3.51}$$

The solution is shown in figure 3.2 Big values of u overtake the small values of u; this is *nonlinear steepening*.

To obtain the solution we can deal with the algebraic result in (3.48) or we can use the geometric construction:

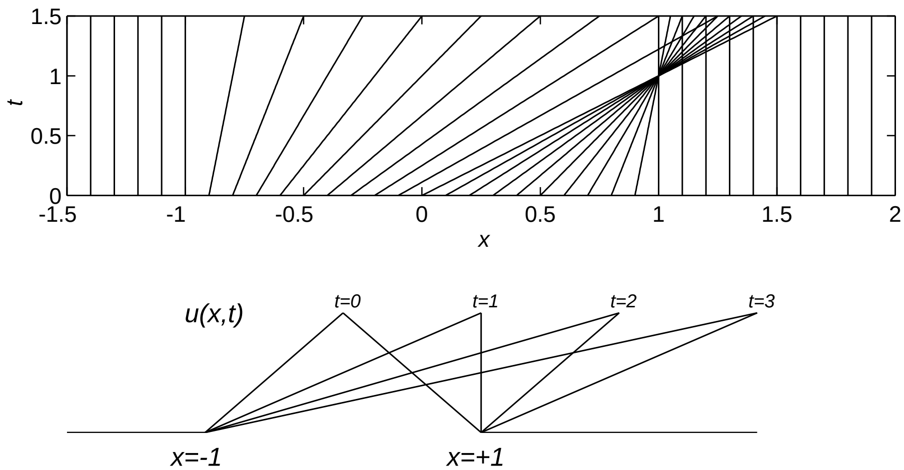

Figure 3.2: The top panel shows the characteristics with the piecewise linear initial condition, F in (3.51). The bottom panel shows the steepening wave. The characteristics first cross, and the solution becomes multivalued, at $(x, t) = (1, 1)$. `kink.eps`

- Draw the graph of the initial condition, $u = F(x)$, in the (x, u)-plane;
- Take each point on the initial curve and slide it sideways a distance $F(x)t$;
- the resulting curve is the graph of the solution $u(x, t)$.

As you see in figure 3.2, this construction gives rise to a multivalued solution when $t > 1$. There is a region of the (x, t)-plane in which there are three values of u.

A witchy initial condition

As an another example, suppose that we want to solve (3.40) with the initial condition

$$F(x) = \frac{1}{1 + x^2} . \tag{3.52}$$

This initial condition is the famous Witch of Agnesi. In this case, from (3.47), the characteristics comprise the family of lines

$$x = \xi + \frac{t}{1 + \xi^2} . \tag{3.53}$$

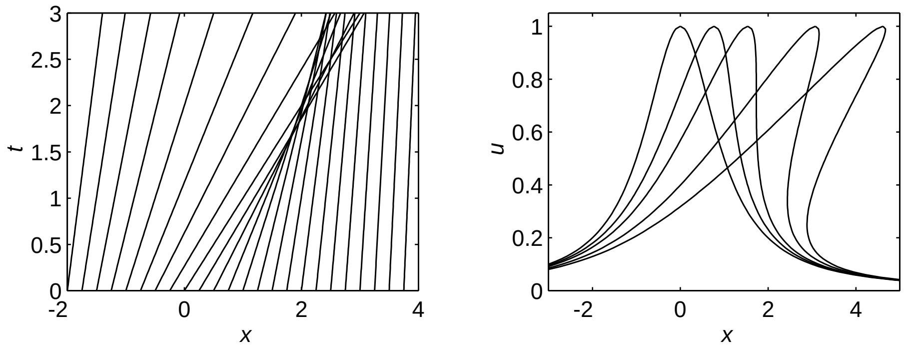

Figure 3.3: The left panel shows the characteristics obtained from (3.53). The right panel shows the steepening wave, calculated using (3.55). The plots are at $t = [0\ 1/2\ 1\ 2\ 3]t_s$, where $t_s = 8\sqrt{3}/9$ is the shock time. advecEqn.eps

Using MATLAB we visualize this family by specifying ξ in (3.53) and then plotting x versus t (see figure 3.3). We see that characteristics intersect, so that there are some points in the (x, t)-plane at which there are three values of u.

To determine $u(x, t)$ we must now eliminate ξ from (3.46) and (3.47):

$$u = \frac{1}{1+\xi^2}, \quad x = \xi + ut \quad \Rightarrow \quad u = \frac{1}{1+(x-ut)^2}. \tag{3.54}$$

Equation (3.54) is a cubic which defines u as an implicit function of x and t. We can visualize the solution without solving this cubic by specifying u and t then solving (3.54) for x in terms of u and t:

$$x_\pm = ut \pm \sqrt{\frac{1-u}{u}}. \tag{3.55}$$

In figure 3.3 we show u as a function of x at fixed times calculated from (3.55) and plotted with MATLAB. The shock first rears its ugly head at $t_s = 8\sqrt{3}/9 \approx 1.54$ when there is a point on the forward face of the pulse at which the slope u_x is infinite (and negative).

Here is the MATLAB script which produces figure 3.3:

```
close all
clc
```

```
%%%%% The characteristic diagram %%%%%
xo=[-2:0.25:4];
t=linspace(0,3);
for xx=xo
    subplot(2,2,1)
    plot(xx+t./(1+xx.^2),t);
    hold on
end
axis([-2 4 0 3])
xlabel('{\it x}')
ylabel('{\it t}')
%%%%%%%% now the right panel %%%%%%%%
ts=8*sqrt(3)/9
u=linspace(0+eps,1);
hold on
for t=[0 ts/2 ts 2*ts 3*ts]
    xp=u*t+sqrt((1-u)./u);
    xm=u*t-sqrt((1-u)./u);
    subplot(2,2,2)
    plot(xm,u,xp,u)
    hold on
    axis([-3 5 0 1.05])
end
xlabel('{\it x}')
ylabel('{\it u}')
```

3.4 Shocks, caustics and multivalued solutions

How did we determine the shock time $t_s = 8\sqrt{3}/9$ in the example of figure 3.3? When $t < t_s$, $u(x,t)$ is a single-valued function of x and the derivative u_x is finite everywhere. But because of nonlinear steepening, u_x becomes infinitely negative at a finite time, known as the shock-time and denoted t_s. When $t > t_s$ the solution is multivalued and there are two locations at which $u_x = \infty$ — see the right hand panel of figure 3.3.

To determine t_s we could go back to our earlier expression for u_x in (3.49) and figure out when and where the infinity first appears. However there is an alternative route which lets us admire some different scenery. Let $v(x,t) \equiv u_x(x,t)$. Differentiating the advection equation (3.40) we have

$$v_t + v^2 + uv_x = 0\,. \tag{3.56}$$

Now we notice that (10.3) evaluated on a characteristic curve is

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -v^2 , \quad \Rightarrow \quad v(\xi, t) = \frac{F'(\xi)}{1 + F'(\xi)t} . \tag{3.57}$$

We see that if $F'(\xi) < 0$ then $v = u_x$ will become infinite in a finite time. The singularity first forms on the characteristic ξ_s which originates at the point of most-negative slope and so

$$t_s = \min_{\forall \xi} \left[-1/F'(\xi) \right] = -1/F'(\xi_s) . \tag{3.58}$$

I like this alternative route because it is obvious from the ODE (3.57) that the slope $v = u_x$ is monotonically decreasing as we move along each characteristic. Also one can calculate t_s *before* solving the PDE.

After the shock forms $u(x, t)$ is a multivalued function: when $t > t_s$ there are three values of u at a single location x. The function $u(x, t)$, viewed as a surface above the (x, t) plane, is folded and the shock location (x_s, t_s) is the "point" of the fold. There are two creases (or caustics) which originate at (x_s, t_s). The caustic curves, $x = x_c(t)$, are located by the condition that $u_x = \infty$, or from (3.49)

$$1 + tF'(\xi_c) = 0 . \tag{3.59}$$

We have to solve the equation above, together with $x_c = \xi_c + F(\xi_c)t$, to determine the caustic location (example below).

In many cases the appearence of a multivalued solution at (x_s, t_s) indicates that the PDE model is failing. For example, if $u(x, t)$ is traffic density it doesn't make sense that there are three different values of u at the same location. In later lectures we discuss how the "solution" of a PDE can be extended using physical arguments to describe evolution once $t > t_s$.

The witch again

In the witchy example:

$$F = \frac{1}{1 + \xi^2} , \qquad F'(\xi) = -2\xi F^2 , \qquad F''(\xi) = 2F^2 \left(4\xi^2 F - 1 \right) . \tag{3.60}$$

The most negative value of F' is at ξ_s, where $F''(\xi_s) = 0$; a short calculation using F'' in (3.60) then gives $\xi_s = 1/\sqrt{3}$. Thus

$$F'(\xi_s) = -3\sqrt{3}/8 , \qquad \text{and from (3.58):} \qquad t_s = 8\sqrt{3}/9 . \tag{3.61}$$

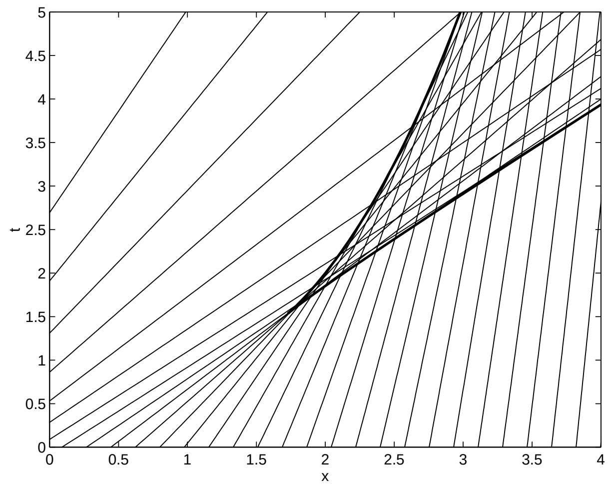

Figure 3.4: The caustic curves of the witch problem. Inside the caustics the characteristics cross and there are three values of u at each (x, t). caust.eps

Can you find the location, x_s, at which the shock forms (see problem 3.18)?

Let's finally complete this witch example by finding the location of the caustics. We have to find the curves in the (x, t) plane along which $u_x = \infty$. In the case of the witch, this leads to the system

$$x_c = \xi_c + \frac{t}{1 + \xi_c^2}, \qquad \frac{2\xi_c}{(1 + \xi_c^2)^2} = \frac{1}{t}. \tag{3.62}$$

The second equation is the condition that $u_x = \infty$, or equivalently $F'(\xi_c) = -1/t$. We must eliminate ξ_c and find x_c as a function of t. This can't be done algebraically. But, once again, it is easy to specify ξ_c, then find t and finally obtain x_c. Here's the MATLAB script which produces figure 3.4:

```
close all
clc
clear
xis=1/(sqrt(3));              %% The xi of the shock.
xi=linspace(0.2*xis,4*xis); %% Specify xi
t=(1+xi.^2).^2./(2*xi);      %% find t
xc=xi+t./(1+xi.^2);          %% find x_caustic
```

```
plot(xc,t,'linewidth',2)
axis([0 4 0 5])
hold on
%%% Now overlay some characteristics
xo=linspace(-2*xis,10*xis,40);
t=linspace(0,5);
for xx=xo
    plot(xx+t./(1+xx.^2),t);
    hold on
end
xlabel('x')
ylabel('t')
```

3.5 Problems

Problem 3.1. Solve PDE:

$$u_x + 2xu_y = xy\,, \quad u(0,y) = 0\,.$$

Solution. Applying the recipe, we find $u(x,y) = \frac{1}{2}yx^2 - \frac{1}{4}x^4$.

Problem 3.2. Is the curve Γ:

$$x = s\,, \quad y = s\,, \quad u = -\frac{1}{1+s}\,,$$

a CC of the PDE $u_x + u_y = u^2$? If it is not, solve the initial value problem; if Γ is a CC, find *all* solutions of the PDE that pass through Γ.

Problem 3.3. *(i)* Show that the solution of

$$yu_x - 2xyu_y = 2xu\,, \quad u(0, 1 \le y \le 2) = y^3\,,$$

is

$$u(x,y) = (y + x^2)^4/y\,.$$

(ii) What is domain of definition of the solution in $y > 0$?

Problem 3.4. Consider the PDE

$$u_x + u_y = 2\,, \qquad u(x,2x) = x^2\,.$$

(i) Verify that the general solution is $u(x,y) = f(y-x) + x + y$, where f is an arbitrary function. Solve the problem by: *(ii)* finding f so that $u(x,y)$ satisfies the initial condition; *(iii)* using the method of characteristics.

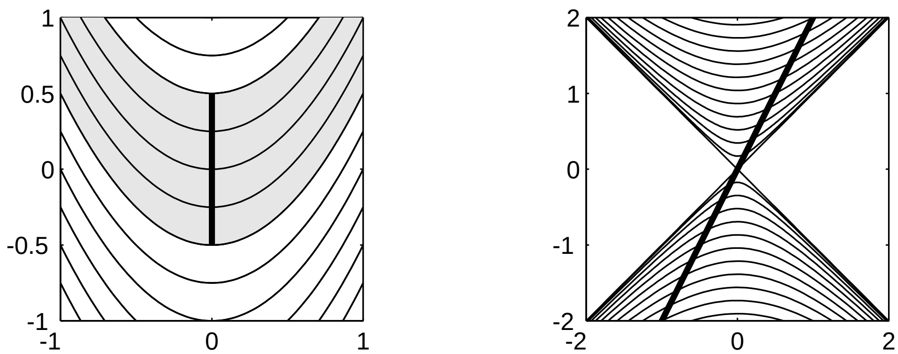

Figure 3.5: The left panel illustrates the solution of problem 3.5 and the right panel shows the solution of problem 3.6. The right panel shows that the domain of definition may be less than the whole plane, even if the DBC has no end points. Moreover, at the origin there is a problem with *existence* of solutions, even at points arbitrarily close to the DC. `domain.eps`

Problem 3.5. Solve the PDE

$$u_x + 2xu_y = xy\,, \quad \text{with } u(0,y) = 1 \text{ on the segment } -\tfrac{1}{2} < y < \tfrac{1}{2}\,.$$

Carefully sketch the region in the (x, y) plane in which this PDE determines $u(x, y)$. That is, find the *domain of definition* of the solution.

Problem 3.6. Solve the PDE:

$$yz_x + xz_y = z - 1\,, \quad z(x, 2x) = x^2 + 2x + 1\,.$$

Solution: We parameterize the initial condition with

$$t = 0: \quad x = s\,, \quad y = 2s\,, \quad z = s^2 + 2s + 1\,. \tag{3.63}$$

Note $J = bx_s - ay_s = -3s$ is zero at $s = 0$.

Notice that the DBC is an infinite line in this example (see the right panel of figure 3.5). Lagrange's equation are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y\,, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = x\,, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = z - 1\,. \tag{3.64}$$

We combine the first two equations and get

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - x = 0\,, \quad \Rightarrow \quad x = A(s)\mathrm{e}^t + B(s)\mathrm{e}^{-t}\,. \tag{3.65}$$

Also

$$y = \frac{\mathrm{d}x}{\mathrm{d}t} = A(s)\mathrm{e}^{t} - B(s)\mathrm{e}^{-t}. \tag{3.66}$$

Now, we let $t = 0$ and apply the initial condition in (3.63):

$$s = A + B, \quad 2s = A - B, \quad \Rightarrow \quad A = \frac{3}{2}s \quad B = \frac{1}{2}s. \tag{3.67}$$

Thus, we now have

$$x(s,t) = \frac{3}{2}s\mathrm{e}^{t} - \frac{1}{2}s\mathrm{e}^{-t}, \quad y(s,t) = \frac{3}{2}s\mathrm{e}^{t} + \frac{1}{2}s\mathrm{e}^{-t}. \tag{3.68}$$

The CBC's obtained from (3.68) are plotted in the right panel of figure 3.5. Notice that none of the CBCs enter the region in which $|y| < |x|$. The domain of definition of this problem is the other region, $|y| > |x|$, which is ventilated by the CBCs which pass through $y = 2x$.

We solve the third equation in (3.64) for z

$$\frac{\mathrm{d}z}{\mathrm{d}t} = z - 1, \quad \Rightarrow \quad z = 1 + C(s)\mathrm{e}^{t}. \tag{3.69}$$

The initial condition in (3.63) now determines $C(s)$ in (3.69):

$$z = 1 + s(s+2)\mathrm{e}^{t}. \tag{3.70}$$

So much for the first two steps of the recipe.

We are now on the home stretch — we eliminate s and t in favor of x and y. If we add and subtract the expressions in (3.68) we get

$$y + x = 3s\mathrm{e}^{t}, \quad y - x = s\mathrm{e}^{-t}. \tag{3.71}$$

Multiplying the equations above together and taking a square root we get

$$s = +\sqrt{\tfrac{1}{3}(y^2 - x^2)}. \tag{3.72}$$

(Why do we take the positive branch of the square root?) Dividing the equations (3.71) and taking another square root we get

$$\mathrm{e}^{t} = \sqrt{\frac{y+x}{3(y-x)}}. \tag{3.73}$$

Armed with (3.72) and (3.73) we can express z in (3.70) in terms of x and y:

$$z(x,y) = 1 + \tfrac{1}{3}(x+y)\left[2 + \sqrt{\tfrac{1}{3}(y^2 - x^2)}\right], \quad \text{BYU}. \tag{3.74}$$

The final step is to check that the expression above solves the PDE■

Problem 3.7. Show that the solution of

$$(x+u)u_x + yu_y = u + y^2\,, \quad u(x,1) = x\,, \quad -\infty < x < \infty\,,$$

is

$$u(x,y) = \frac{x-y^2}{1+\ln y} + y^2\,.$$

What is the domain of definition of $u(x,y)$?

Problem 3.8. Solve

$$xu_x + yuu_y = -xy\,, \qquad u(x,y) = 5 \text{ on } xy = 1\,.$$

Problem 3.9. Solve

$$yu_x + x^3u_y = x^3y\,, \qquad u(x,x^2) = x^4\,.$$

Problem 3.10. Show that the solution of

$$x^3u_x = u_y\,, \qquad u(x,0) = \frac{1}{1+x^2}\,,$$

is

$$u(x,y) = \frac{1-2x^2y}{1+x^2-2x^2y}\,.$$

Show that the solution is not defined in $y > 1/2x^2$ even though the data is prescribed for $-\infty < x < \infty$.

Problem 3.11. Show that the initial value problem

$$(y-x)u_x - (y+x)u_y = 0\,, \quad u(x,0) = f(x)\,, \quad x > 0\,,$$

has no solution if $f(x)$ is an arbitrary function.

Problem 3.12. Consider the initial-value problems:

$$u_x - u_y = u\,, \quad u(x,-x+x^2) = \mathrm{e}^x\,,$$
$$u_x - u_y = u\,, \quad u(x,-x+x^2) = 1\,.$$

If the solution exists, find it and the domain of existence. If there is no solution, explain why.

Problem 3.13. Show that the initial value problem

$$yu_x + xu_y = pu \qquad u(x,x) = f(x)\,,$$

has no solution unless $f(x) = ax^p$. If f has this form, find the solution.

Problem 3.14. *(i)* Find a solution analogous to (3.48) of the PDE

$$u_t + c(u)u_x = 0\,, \quad u(x,0) = F(x)\,.$$

(ii) Check your answer by substitution. *(iii)* Considering the special case $c = u^2$ and $F = x$, show that once $t > 0$ there are either two real values of $u(x,t)$ or no real values of $u(x,t)$ at each point in spacetime. Locate the curve in the (x,t) plane which separates these two behaviours.

Problem 3.15. *(i)* Find an implicit solution, analogous to (3.48), of

$$u_t + uu_x = -\alpha u\,, \quad u(x,0) = \frac{1}{1+x^2}\,.$$

Make sure your answer reduces to (3.48) if $\alpha \to 0$. Draw a figure, like 3.3, showing both characteristics and snapshots of $u(x,t)$. (Use several values of α so that you understand how this parameter changes the solution). *(ii)* Calculate t_s as a function of α. *(iii)* Find the smallest value of the damping α which is sufficient to prevent shock formation.

Problem 3.16. *(i)* Does the PDE

$$u_t + uu_x = 0\,, \quad u(x,0) = \frac{e^x}{1+e^x}\,.$$

form a shock? If so, calculate t_s without solving the PDE. *(ii)* Find an implicit solution for $u(x,t)$ and use MATLAB to graph this solution.

Solution. The geometric construction described following (3.51) makes it clear that this initial condition does not form a shock. (Also notice that the initial slope, $u_x(x,0)$, is positive everywhere and so according to (3.58) the shock time, t_s, is in the past.) In part *(ii)*, we quickly find

$$\xi = x - \frac{e^\xi}{1+e^\xi}t \qquad u = \frac{e^\xi}{1+e^\xi}\,.$$

Fiddling about with the second equation above, we can express ξ as an explicit function of u: $\xi = \ln[u/(u-1)]$. Next, we can eliminate ξ and give an equation which defines u implicitly as a function of x and t:

$$\ln\left(\frac{u}{u-1}\right) = x - ut\,.$$

We can't solve this monster for $u(x,t)$ explicitly. But to graph $u(x,t)$, just as in the right panel of figure 3.3, we don't need to...

Problem 3.17. *(i)* Does the PDE

$$u_t + uu_x = 0\,, \quad u(x,0) = \frac{\mathrm{e}^{-x}}{1+\mathrm{e}^{-x}}\,.$$

form a shock? If so, calculate t_s without solving the PDE. *(ii)* Find an implicit solution for $u(x,t)$ and use MATLAB to graph this solution.

Problem 3.18. In the witchy example we showed in (3.60) and (3.61) that the time at which the shock forms is $t_s = 8\sqrt{3}/9$. Find the location, x_s, at which the shock forms.

Problem 3.19. Find an expression for t_s, analogous to (3.58), for the PDE

$$u_t + u^2 u_x = 0\,, \quad u(x,0) = F(x)\,.$$

Also give a simple formula for x_s.

Problem 3.20. Consider

$$u_t + u^3 u_x = 0\,, \quad u(x,0) = \sin x\,.$$

At what time, t_s, and location, x_s, does the solution $u(x,t)$ first become singular?