Exercise 10.1.3

Show that the function $y(x)$ defined by Eq. (10.26) satisfies the initial-value problem defined by Eq. (10.24) and its initial conditions $y(0) = y'(0) = 0$.

Solution

Eq. (10.26) in the text is

$$y(x) = \int_0^x \sin(x-t) f(t)\, dt. \tag{10.26}$$

The aim here is to show that it satisfies

$$\frac{d^2y}{dx^2} + y = f(x), \qquad y(0) = 0,\ y'(0) = 0. \tag{10.24}$$

Differentiate $y(x)$ with respect to x by using the Leibnitz rule.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \int_0^x \sin(x-t) f(t)\, dt \\
&= \int_0^x \frac{\partial}{\partial x} \sin(x-t) f(t)\, dt + \sin(x-x) f(x) \cdot 1 - \sin(x-0) f(0) \cdot 0 \\
&= \int_0^x \cos(x-t) f(t)\, dt
\end{aligned}$$

Differentiate dy/dx with respect to x once more.

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx} \int_0^x \cos(x-t) f(t)\, dt \\
&= \int_0^x \frac{\partial}{\partial x} \cos(x-t) f(t)\, dt + \cos(x-x) f(x) \cdot 1 - \cos(x-0) f(0) \cdot 0 \\
&= \int_0^x [-\sin(x-t)] f(t)\, dt + f(x) \\
&= -\int_0^x \sin(x-t) f(t)\, dt + f(x) \\
&= -y(x) + f(x)
\end{aligned}$$

Therefore, the integral solution satisfies the ODE. Finally, check that the initial conditions are satisfied.

$$\begin{aligned}
y(0) &= \int_0^0 \sin(0-t) f(t)\, dt = 0 \\
\frac{dy}{dx}(0) &= \int_0^0 \cos(0-t) f(t)\, dt = 0
\end{aligned}$$