Exercise 10.1.5

Construct the Green's function for

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (k^2x^2 - 1)y = 0,$$

subject to the boundary conditions $y(0) = 0$, $y(1) = 0$.

Solution

The Green's function for an operator $\mathcal{L}$ satisfies

$$\mathcal{L}G = \delta(x - t).$$

Part (a)

For the operator $\mathcal{L} = x^2(d^2/dx^2) + x(d/dx) + k^2x^2 - 1$, this equation becomes

$$x^2 \frac{d^2G}{dx^2} + x\frac{dG}{dx} + (k^2x^2 - 1)G = \delta(x - t). \tag{1}$$

If $x \neq t$, then the right side is zero.

$$x^2 \frac{d^2G}{dx^2} + x\frac{dG}{dx} + (k^2x^2 - 1)G = 0, \quad x \neq t$$

The general solution can be written in terms of first-order Bessel functions of the first and second kind. Different constants are needed for $x < t$ and for $x > t$.

$$G(x,t) = \begin{cases} C_1J_1(kx) + C_2Y_1(kx) & \text{if } 0 \leq x < t \\ C_3J_1(kx) + C_4Y_1(kx) & \text{if } t < x \leq 1 \end{cases}$$

Four conditions are needed to determine these four constants. Two of them are obtained from the provided boundary conditions.

$$\begin{aligned} G(0,t) &= 0 & \Rightarrow \quad & C_2 = 0 \\ G(1,t) &= C_3J_1(k) + C_4Y_1(k) = 0 & \rightarrow \quad & C_4 = -C_3\frac{J_1(k)}{Y_1(k)} \end{aligned}$$

As a result, the Green's function becomes

$$G(x,t) = \begin{cases} C_1J_1(kx) & \text{if } 0 \leq x < t \\ C_3J_1(kx) - C_3\frac{J_1(k)}{Y_1(k)}Y_1(kx) & \text{if } t < x \leq 1 \end{cases}.$$

The third condition comes from the fact that the Green's function must be continuous at $x = t$: $G(t-,t) = G(t+,t)$.

$$\begin{aligned} C_1J_1(kt) &= C_3J_1(kt) - C_3\frac{J_1(k)}{Y_1(k)}Y_1(kt) \\ &= C_3\frac{J_1(kt)Y_1(k) - J_1(k)Y_1(kt)}{Y_1(k)} \end{aligned}$$

Solve for C_1.

$$C_1 = C_3 \frac{J_1(kt)Y_1(k) - J_1(k)Y_1(kt)}{J_1(kt)Y_1(k)} \tag{2}$$

The fourth and final condition is obtained from the defining equation of the Green's function, equation (1).

$$x^2 \frac{d^2G}{dx^2} + x\frac{dG}{dx} + (k^2x^2 - 1)G = \delta(x - t)$$

Integrate both sides with respect to x from $t-$ to $t+$.

$$\int_{t-}^{t+} \left[x^2 \frac{d^2G}{dx^2} + x\frac{dG}{dx} + (k^2x^2 - 1)G\right] dx = \int_{t-}^{t+} \delta(x - t)\, dx$$

$$\int_{t-}^{t+} x^2 \frac{d^2G}{dx^2}\, dx + \int_{t-}^{t+} x\frac{dG}{dx}\, dx + \underbrace{\int_{t-}^{t+} (k^2x^2 - 1)G\, dx}_{=0} = \underbrace{\int_{t-}^{t+} \delta(x - t)\, dx}_{=1}$$

$$x^2 \frac{dG}{dx}\bigg|_{t-}^{t+} - \int_{t-}^{t+} (2x)\frac{dG}{dx}\, dx + \int_{t-}^{t+} x\frac{dG}{dx}\, dx = 1$$

$$x^2 \frac{dG}{dx}\bigg|_{t-}^{t+} - \int_{t-}^{t+} x\frac{dG}{dx}\, dx = 1$$

$$x^2 \frac{dG}{dx}\bigg|_{t-}^{t+} - xG(x,t)\bigg|_{t-}^{t+} + \int_{t-}^{t+} (1)G(x,t)\, dx = 1$$

$$x^2 \frac{dG}{dx}\bigg|_{t-}^{t+} - t[\underbrace{G(t+,t) - G(t-,t)}_{=0}] + \underbrace{\int_{t-}^{t+} G(x,t)\, dx}_{=0} = 1$$

$$t^2 \left[\frac{dG}{dx}(t+,t) - \frac{dG}{dx}(t-,t)\right] = 1$$

Divide both sides by t^2.

$$\frac{dG}{dx}(t+,t) - \frac{dG}{dx}(t-,t) = \frac{1}{t^2}$$

$$\frac{d}{dx}\left[C_3J_1(kx) - C_3\frac{J_1(k)}{Y_1(k)}Y_1(kx)\right]\bigg|_{x=t} - \frac{d}{dx}\left[C_1J_1(kx)\right]\bigg|_{x=t} = \frac{1}{t^2}$$

$$C_3\frac{k}{2}\frac{Y_1(k)[J_0(kt) - J_2(kt)] - J_1(k)[Y_0(kt) - Y_2(kt)]}{Y_1(k)} - C_1\frac{k}{2}[J_0(kt) - J_2(kt)] = \frac{1}{t^2}$$

Multiply both sides by $2/k$ and substitute equation (2) for C_1.

$$C_3\frac{Y_1(k)[J_0(kt) - J_2(kt)] - J_1(k)[Y_0(kt) - Y_2(kt)]}{Y_1(k)} - C_3\frac{J_1(kt)Y_1(k) - J_1(k)Y_1(kt)}{J_1(kt)Y_1(k)}[J_0(kt) - J_2(kt)] = \frac{2}{kt^2}$$

$$C_3\frac{J_0(kt)J_1(k)Y_1(kt) - J_1(k)J_1(kt)Y_0(kt) - J_1(k)J_2(kt)Y_1(kt) + J_1(k)J_1(kt)Y_2(kt)}{J_1(kt)Y_1(k)} = \frac{2}{kt^2}$$

$$C_3J_1(k)\frac{Y_1(kt)[J_0(kt) - J_2(kt)] - J_1(kt)[Y_0(kt) - Y_2(kt)]}{J_1(kt)Y_1(k)} = \frac{2}{kt^2}$$

Solve for C_3.

$$C_3 = \frac{2}{kt^2}\frac{1}{J_1(k)}\frac{J_1(kt)Y_1(k)}{Y_1(kt)[J_0(kt) - J_2(kt)] - J_1(kt)[Y_0(kt) - Y_2(kt)]}$$

Use equation (2) to get C_1.

$$\begin{aligned} C_1 &= C_3 \frac{J_1(kt)Y_1(k) - J_1(k)Y_1(kt)}{J_1(kt)Y_1(k)} \\ &= \frac{2}{kt^2}\frac{1}{J_1(k)}\frac{J_1(kt)Y_1(k) - J_1(k)Y_1(kt)}{Y_1(kt)[J_0(kt) - J_2(kt)] - J_1(kt)[Y_0(kt) - Y_2(kt)]} \end{aligned}$$

The Green's function for $\mathcal{L} = x^2(d^2/dx^2) + x(d/dx) + k^2x^2 - 1$ subject to the provided boundary conditions is then

$$G(x,t) = \begin{cases} \dfrac{2}{kt^2}\dfrac{J_1(kx)}{J_1(k)}\dfrac{J_1(kt)Y_1(k) - J_1(k)Y_1(kt)}{Y_1(kt)[J_0(kt) - J_2(kt)] - J_1(kt)[Y_0(kt) - Y_2(kt)]} & \text{if } 0 \le x < t \\ \dfrac{2}{kt^2}\dfrac{J_1(kt)}{J_1(k)}\dfrac{J_1(kx)Y_1(k) - J_1(k)Y_1(kx)}{Y_1(kt)[J_0(kt) - J_2(kt)] - J_1(kt)[Y_0(kt) - Y_2(kt)]} & \text{if } t < x \le 1 \end{cases}.$$

This formula can be simplified by using the identity,

$$Y_1(kt)[J_0(kt) - J_2(kt)] - J_1(kt)[Y_0(kt) - Y_2(kt)] = -\frac{4}{\pi kt}.$$

Therefore,

$$G(x,t) = \begin{cases} \dfrac{\pi}{2t}\dfrac{J_1(kx)}{J_1(k)}[J_1(k)Y_1(kt) - J_1(kt)Y_1(k)] & \text{if } 0 \le x < t \\ \dfrac{\pi}{2t}\dfrac{J_1(kt)}{J_1(k)}[J_1(k)Y_1(kx) - J_1(kx)Y_1(k)] & \text{if } t < x \le 1 \end{cases}.$$