Exercise 7.2.10

A certain differential equation has the form

\[ f(x) \, dx + g(x)h(y) \, dy = 0, \]

with none of the functions \( f(x), g(x), h(y) \) identically zero. Show that a necessary and sufficient condition for this equation to be exact is that \( g(x) = \text{constant} \).

Solution

Suppose first that the ODE is exact. Then there exists a potential function \( \varphi = \varphi(x, y) \) that satisfies

\[
\frac{\partial \varphi}{\partial x} = f(x) \tag{1} \\
\frac{\partial \varphi}{\partial y} = g(x)h(y). \tag{2}
\]

Substitute these formulas into the ODE.

\[
\frac{\partial \varphi}{\partial x} \, dx + \frac{\partial \varphi}{\partial y} \, dy = 0
\]

On the left is how the differential of \( \varphi \) is defined.

\[ d\varphi = 0 \]

Integrate both sides.

\[ \varphi(x, y) = C \]

The solution to the ODE is found then by solving equations (1) and (2) for \( \varphi \). Start by integrating both sides of equation (1) partially with respect to \( x \).

\[ \varphi(x, y) = \int_{r}^{x} f(r) \, dr + F(y) \]

\( F(y) \) here is an arbitrary function of \( y \), and the lower limit of integration is arbitrary as well. Differentiate both sides with respect to \( y \).

\[
\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y} \left[ \int_{r}^{x} f(r) \, dr + F(y) \right] \\
= \frac{\partial}{\partial y} \int_{r}^{x} f(r) \, dr + F'(y) \\
= F'(y)
\]

Comparing this formula for \( \frac{\partial \varphi}{\partial y} \) to equation (2), we see that

\[ F'(y) = g(x)h(y). \]

In order to integrate both sides with respect to \( y \) and solve for \( F(y) \), the function \( g(x) \) must be a constant.

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In order for the ODE,

\[ f(x) \, dx + g(x)h(y) \, dy = 0, \]

to be exact, it must satisfy

\[
\frac{\partial}{\partial y} [f(x)] = \frac{\partial}{\partial x} [g(x)h(y)]
\]

\[ 0 = h(y) \frac{dg}{dx}. \]

Since \( h(y) \neq 0 \), divide both sides by \( h(y) \).

\[ \frac{dg}{dx} = 0 \]

Integrate both sides with respect to \( x \).

\[ g(x) = \text{constant} \]

Therefore, a necessary and sufficient condition for this equation to be exact is that \( g(x) = \text{constant} \).