Exercice 7.2.13

Radioactive nuclei decay according to the law

\[
\frac{dN}{dt} = -\lambda N,
\]

\(N\) being the concentration of a given nuclide and \(\lambda\), the particular decay constant. In a radioactive series of two different nuclides, with concentrations \(N_1(t)\) and \(N_2(t)\), we have

\[
\frac{dN_1}{dt} = -\lambda_1 N_1,
\]

\[
\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2.
\]

Find \(N_2(t)\) for the conditions \(N_1(0) = N_0\) and \(N_2(0) = 0\).

**Solution**

Begin by solving the ODE for \(N_1(t)\). Divide both sides by \(N_1\).

\[
\frac{dN_1}{dt} N_1 = -\lambda_1
\]

The left side can be written as the derivative of a logarithm by the chain rule.

\[
\frac{d}{dt}(\ln N_1) = -\lambda_1
\]

Integrate both sides with respect to \(t\).

\[
\ln N_1 = -\lambda_1 t + C_1
\]

Exponentiate both sides.

\[
N_1(t) = e^{-\lambda_1 t + C_1}
\]

\[
= e^{-\lambda_1 t} e^{C_1}
\]

Apply the initial condition \(N_1(0) = N_0\) to determine \(e^{C_1}\).

\[
N(0) = e^{-\lambda_1(0)} e^{C_1} \quad \rightarrow \quad N_0 = e^{C_1}
\]

So then

\[
N_1(t) = N_0 e^{-\lambda_1 t}.
\]

Substitute this formula for \(N_1\) into the second ODE involving \(N_2\).

\[
\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2
\]

\[
\frac{dN_2}{dt} = \lambda_1 N_0 e^{-\lambda_1 t} - \lambda_2 N_2
\]

Bring \(\lambda_2 N_2\) to the left side.

\[
\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_0 e^{-\lambda_1 t}
\]

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This is a first-order linear ODE, so it can be solved by multiplying both sides by an integrating factor $I$.

$$I = \exp \left( \int \lambda_2 \, ds \right) = e^{\lambda_2 t}$$

Proceed with the multiplication.

$$e^{\lambda_2 t} \frac{dN_2}{dt} + \lambda_2 e^{\lambda_2 t} N_2 = \lambda_1 e^{\lambda_2 t} N_0 e^{-\lambda_1 t}$$

The left side can now be written as a derivative by the product rule. Combine the exponential functions on the right side.

$$\frac{d}{dt}(e^{\lambda_2 t} N_2) = N_0 \lambda_1 e^{(\lambda_2 - \lambda_1) t}$$ (1)

Suppose first that $\lambda_1 \neq \lambda_2$. Integrate both sides with respect to $t$.

$$e^{\lambda_2 t} N_2 = N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1) t} + C_2$$

Divide both sides by $e^{\lambda_2 t}$.

$$N_2(t) = N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t}$$

Apply the initial condition $N_2(0) = 0$ now to determine $C_2$.

$$N_2(0) = N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 (0)} + C_2 e^{-\lambda_2 (0)}$$

$$0 = N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1} + C_2$$

$$C_2 = -N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1}$$

Plug this back into the general solution for $N_2(t)$.

$$N_2(t) = N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} - N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}$$

$$= N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}), \quad \lambda_1 \neq \lambda_2$$

Suppose secondly that $\lambda_1 = \lambda_2$. Then equation (1) becomes

$$\frac{d}{dt}(e^{\lambda_2 t} N_2) = N_0 \lambda_2.$$ 

Integrate both sides with respect to $t$.

$$e^{\lambda_2 t} N_2 = N_0 \lambda_2 t + C_3$$

Divide both sides by $e^{\lambda_2 t}$.

$$N_2(t) = e^{-\lambda_2 t} (N_0 \lambda_2 t + C_3)$$
Apply the initial condition $N_2(0) = 0$ to determine $C_3$.

$$N_2(0) = C_3 = 0$$

So then

$$N_2(t) = N_0\lambda_2 t e^{-\lambda_2 t}, \quad \lambda_1 = \lambda_2.$$ 

Therefore,

$$N_1(t) = N_0 e^{-\lambda_1 t}$$

$$N_2(t) = \begin{cases} 
N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}) & \text{if } \lambda_1 \neq \lambda_2 \\
N_0\lambda_2 t e^{-\lambda_2 t} & \text{if } \lambda_1 = \lambda_2
\end{cases}.$$