Exercise 7.2.15

In the linear homogeneous differential equation

$$\frac{dv}{dt} = -av$$

the variables are separable. When the variables are separated, the equation is exact. Solve this differential equation subject to $v(0) = v_0$ by the following three methods:

(a) Separating variables and integrating.

(b) Treating the separated variable equation as exact.

(c) Using the result for a linear homogeneous differential equation.

ANS. $v(t) = v_0 e^{-at}$.

Solution

Method (a)

Separate variables in the ODE.

$$\frac{dv}{v} = -a\,dt$$

Integrate both sides.

$$\int \frac{dv}{v} = \int -a\,dt$$

Evaluate the integrals.

$$\ln|v| = -at + C$$

Exponentiate both sides.

$$\begin{aligned} |v| &= e^{-at+C} \\ &= e^{-at}e^{C} \end{aligned}$$

Remove the absolute value sign on the left by placing $\pm$ on the right side.

$$v(t) = \pm e^{C} e^{-at}$$

Use a new constant A for $\pm e^{C}$.

$$v(t) = Ae^{-at}$$

Apply the initial condition $v(0) = v_0$ to determine A.

$$v(0) = Ae^{-a(0)}$$

$$v_0 = A$$

Therefore,

$$v(t) = v_0 e^{-at}.$$

Method (b)

Consider the separated variable equation in the previous part.

$$\frac{dv}{v} = -a\,dt$$

Bring $a\,dt$ to the left side.

$$\frac{dv}{v} + a\,dt = 0 \tag{1}$$

Since the ODE is exact, that is,

$$\frac{\partial}{\partial t}\left(\frac{1}{v}\right) = \frac{\partial}{\partial v}(a)$$
$$0 = 0,$$

there exists a potential function $\varphi = \varphi(x,t)$ that satisfies

$$\frac{\partial \varphi}{\partial v} = \frac{1}{v} \tag{2}$$

$$\frac{\partial \varphi}{\partial t} = a. \tag{3}$$

In terms of it, equation (1) becomes

$$\frac{\partial \varphi}{\partial v}\,dv + \frac{\partial \varphi}{\partial t}\,dt = 0.$$

On the left is how the differential of φ is defined.

$$d\varphi = 0$$

Integrate both sides.

$$\varphi(x,t) = C_1$$

The solution to the ODE is found then by solving equations (2) and (3) for φ. Integrate both sides of equation (2) partially with respect to t.

$$\varphi(x,t) = at + f(v)$$

Differentiate it with respect to v.

$$\frac{\partial \varphi}{\partial v} = f'(v)$$

Comparing this to equation (1), we see that

$$f'(v) = \frac{1}{v}.$$

Integrate both sides with respect to v.

$$f(v) = \ln|v| + C_2$$

Consequently, the potential function is

$$\varphi(x,t) = at + \ln|v| + C_2,$$

and the solution is

$$at + \ln v + C_2 = C_1 \quad \rightarrow \quad at + \ln|v| = C_3,$$

where a new constant C_3 is used for $C_1 - C_2$. Solve for v.

$$\ln|v| = C_3 - at$$

Exponentiate both sides.

$$\begin{aligned}|v| &= e^{C_3 - at} \\ &= e^{C_3} e^{-at}\end{aligned}$$

Remove the absolute value sign by placing $\pm$ on the right side.

$$v(t) = \pm e^{C_3} e^{-at}$$

Use a new constant C_4 for $\pm e^{C_3}$.

$$v(t) = C_4 e^{-at}$$

Now apply the initial condition $v(0) = v_0$ to determine C_4.

$$v(0) = C_4 e^{-a(0)} \quad \rightarrow \quad v_0 = C_4$$

Therefore,

$$v(t) = v_0 e^{-at}.$$

Method (c)

Bring av to the left side.

$$\frac{dv}{dt} + av = 0$$

This is a linear first-order ODE for v, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t a\,ds\right) = e^{at}$$

Proceed with the multiplication.

$$e^{at}\frac{dv}{dt} + ae^{at}v = 0$$

The left side can be written as $d/dt(Iv)$ by the product rule.

$$\frac{d}{dt}(e^{at}v) = 0$$

Integrate both sides with respect to t.

$$e^{at}v = C_5$$

Divide both sides by e^{at}.

$$v(t) = C_5 e^{-at}$$

Apply the initial condition $v(0) = v_0$ to determine C_5.

$$v(0) = C_5 e^{-a(0)}$$

$$v_0 = C_5$$

Therefore,

$$v(t) = v_0 e^{-at}.$$