Exercise 7.2.2

The Laplace transform of Bessel’s equation \((n = 0)\) leads to

\[(s^2 + 1)f'(s) + sf(s) = 0.\]

Solve for \(f(s)\).

Solution

Bring \(sf(s)\) to the right side.

\[(s^2 + 1)f'(s) = -sf(s)\]

Divide both sides by \(s^2 + 1\).

\[f'(s) = -\frac{s}{s^2 + 1}f(s)\]

Divide both sides by \(f(s)\).

\[\frac{f'(s)}{f(s)} = -\frac{s}{s^2 + 1}\]

The left side can be written as the derivative of a logarithm by the chain rule.

\[\frac{d}{ds} \ln f(s) = -\frac{s}{s^2 + 1}\]

Integrate both sides with respect to \(s\).

\[\ln f(s) = -\int \frac{r}{r^2 + 1} \, dr + C\]

To evaluate the integral, let \(u = r^2 + 1\). Then \(du = 2r \, dr\).

\[\ln f(s) = -\frac{1}{2} \int \frac{1}{u} \, du + C\]

\[= -\frac{1}{2} \ln u \bigg|^{s^2 + 1} + C\]

\[= -\frac{1}{2} \ln(s^2 + 1) + C\]

\[= \frac{1}{2} \ln(s^2 + 1)^{-1/2} + C\]

Exponentiate both sides.

\[e^{\ln f(s)} = e^{\ln(s^2+1)^{-1/2} + C}\]

\[f(s) = e^{\ln(s^2+1)^{-1/2} + C} e^{C}\]

\[= (s^2 + 1)^{-1/2} e^{C}\]

\[= e^{C} / (s^2 + 1)^{1/2}\]

Therefore, using a new constant \(A\) for \(e^{C}\),

\[f(s) = \frac{A}{(s^2 + 1)^{1/2}}.\]

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