Exercise 7.2.7

The differential equation

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

is **exact**. Show that its solution is of the form

$$\varphi(x,y) = \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x_0,y)\,dy = \text{constant}.$$

Solution

The ODE is exact, so there exists a potential function $\varphi = \varphi(x,y)$ that satisfies

$$\frac{\partial \varphi}{\partial x} = P(x,y) \tag{1}$$

$$\frac{\partial \varphi}{\partial y} = Q(x,y). \tag{2}$$

With these two relationships, the ODE can be written as

$$\frac{\partial \varphi}{\partial x}\,dx + \frac{\partial \varphi}{\partial y}\,dy = 0.$$

The left side is how the differential of φ is defined.

$$d\varphi = 0$$

Integrate both sides.

$$\varphi(x,y) = C \tag{3}$$

The solution to the ODE is found then by solving equations (1) and (2) for φ. Start by integrating both sides of equation (1) partially with respect to x.

$$\varphi(x,y) = \int^{x} P(r,y)\,dr + f(y)$$

Here $f(y)$ is an arbitrary function of y. Because of it, the lower limit of integration is arbitrary and can be set to any constant, say x_0.

$$\varphi(x,y) = \int_{x_0}^{x} P(r,y)\,dr + f(y)$$

Differentiate both sides with respect to y.

$$\begin{aligned}
\frac{\partial \varphi}{\partial y} &= \frac{\partial}{\partial y}\int_{x_0}^{x} P(r,y)\,dr + f'(y) \\
&= \int_{x_0}^{x} \left.\frac{\partial P}{\partial y}\right|_{x=r} dr + f'(y) \\
&= \int_{x_0}^{x} \left.\left[\frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right)\right]\right|_{x=r} dr + f'(y) \\
&= \int_{x_0}^{x} \left.\left[\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right)\right]\right|_{x=r} dr + f'(y) \\
&= \int_{x_0}^{x} \left.\frac{\partial Q}{\partial x}\right|_{x=r} dr + f'(y) \\
&= Q(x,y) - Q(x_0,y) + f'(y)
\end{aligned}$$

Comparing this formula for $\partial\varphi/\partial y$ to equation (2), we see that

$$-Q(x_0, y) + f'(y) = 0.$$

The arbitrary function f can be determined.

$$f'(y) = Q(x_0, y)$$

Integrate both sides with respect to y.

$$f(y) = \int^y Q(x_0, s)\,ds + C_1$$

Because C_1 is arbitrary, the lower limit of integration is arbitrary as well. Set it to y_0.

$$f(y) = \int_{y_0}^y Q(x_0, s)\,ds + C_1$$

φ is then

$$\varphi(x, y) = \int_{x_0}^x P(r, y)\,dr + \int_{y_0}^y Q(x_0, s)\,ds + C_1.$$

Therefore, by equation (3), the solution to the exact ODE is

$$\int_{x_0}^x P(r, y)\,dr + \int_{y_0}^y Q(x_0, s)\,ds = \text{constant}.$$