Exercise 7.2.9

Prove that Eq. (7.12) is exact in the sense of Eq. (7.9), provided that $\alpha(x)$ satisfies Eq. (7.14).

Solution

Eq. (7.12) is on page 336.

$$\alpha(x) \frac{dy}{dx} + \alpha(x)p(x)y = \alpha(x)q(x) \quad (7.12)$$

Here $\alpha(x)$ is an integrating factor that’s been multiplied by both sides of a linear first-order inhomogeneous ODE. It satisfies

$$\frac{d\alpha}{dx} = \alpha(x)p(x). \quad (7.14)$$

Bring all terms in Eq. (7.12) to the left side.

$$\alpha(x)\frac{dy}{dx} + \alpha(x)p(x)y - \alpha(x)q(x) = 0$$

Multiply both sides by $dx$.

$$[\alpha(x)p(x)y - \alpha(x)q(x)] \, dx + [\alpha(x)] \, dy = 0$$

For this ODE to be exact, it must satisfy the condition in Eq. (7.9).

$$\frac{\partial}{\partial y} \left[ \alpha(x)p(x)y - \alpha(x)q(x) \right] = \frac{\partial}{\partial x} \left[ \alpha(x) \right]$$

$$\alpha(x) p(x) = \frac{d\alpha}{dx}$$

$$\alpha(x)p(x) = \alpha(x)p(x)$$

Because this condition is satisfied, the ODE in Eq. (7.12) is exact, and there exists a potential function $\varphi = \varphi(x, y)$ such that

$$\frac{\partial \varphi}{\partial x} = \alpha(x)p(x)y - \alpha(x)q(x)$$

$$\frac{\partial \varphi}{\partial y} = \alpha(x).$$