Exercise 7.3.2

Find the general solutions to the following ODEs. Write the solutions in forms that are entirely real (i.e., that contain no complex quantities).

\[ y''' - 2y'' + y' - 2y = 0. \]

Solution

Because this is a linear homogeneous ODE and all the coefficients on the left side are constant, the solutions for it are of the form \( y = e^{rt} \).

\[ y = e^{rt} \rightarrow y' = re^{rt} \rightarrow y'' = r^2 e^{rt} \rightarrow y''' = r^3 e^{rt} \]

Substitute these formulas into the ODE.

\[ r^3 e^{rt} - 2r^2 e^{rt} + re^{rt} - 2e^{rt} = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^3 - 2r^2 + r - 2 = 0 \]

Solve for \( r \).

\[ (r - 2)(r^2 + 1) = 0 \]

\[ (r - 2)(r + i)(r - i) = 0 \]

\[ r = \{2, -i, i\} \]

Three solutions to the ODE are \( y = e^{2t} \) and \( y = e^{-it} \) and \( y = e^{it} \). By the principle of superposition, the general solution is a linear combination of these three. Therefore,

\[ y(t) = C_1 e^{2t} + C_2 e^{-it} + C_3 e^{it} \]

\[ = C_1 e^{2t} + C_2 [\cos(-t) + i \sin(-t)] + C_3 (\cos t + i \sin t) \]

\[ = C_1 e^{2t} + C_2 (\cos t - i \sin t) + C_3 (\cos t + i \sin t) \]

\[ = C_1 e^{2t} + C_2 \cos t - iC_2 \sin t + C_3 \cos t + iC_3 \sin t \]

\[ = C_1 e^{2t} + (C_2 + C_3) \cos t + (-iC_2 + iC_3) \sin t \]

\[ = C_1 e^{2t} + C_4 \cos t + C_5 \sin t. \]