Exercise 7.4.1

Show that Legendre's equation has regular singularities at $x = -1, 1,$ and $\infty$.

Solution

Legendre’s equation is a second-order linear homogeneous ODE.

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0$$

Divide both sides by $1-x^2$ so that the coefficient of $y''$ is 1.

$$y'' - \frac{2x}{1-x^2}y' + \frac{l(l+1)}{1-x^2}y = 0$$

There are singular points where the denominators are equal to zero: $x = \pm 1$. $x = -1$ is regular because the following limits are finite.

$$\lim_{x \to -1} (x+1)\left(-\frac{2x}{1-x^2}\right) = \lim_{x \to -1} \left(-\frac{2x}{1-x}\right) = 1$$

$$\lim_{x \to -1} \frac{(x+1)^2l(l+1)}{1-x^2} = \lim_{x \to -1} \frac{l(l+1)(x+1)}{1-x} = 0$$

$x = 1$ is regular for the same reason.

$$\lim_{x \to 1} (x-1)\left(-\frac{2x}{1-x^2}\right) = \lim_{x \to 1} \left(\frac{2x}{1+x}\right) = 1$$

$$\lim_{x \to 1} \frac{(x-1)^2l(l+1)}{1-x^2} = \lim_{x \to 1} \frac{l(l+1)(1-x)}{1+x} = 0$$

In order to investigate the behavior at $x = \infty$, make the substitution,

$$x = \frac{1}{z},$$

in Legendre’s equation.

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0 \quad \rightarrow \quad \left(1 - \frac{1}{z^2}\right) y'' - \frac{2}{z} y' + l(l+1)y = 0$$

Use the chain rule to find what the derivatives of $y$ are in terms of this new variable.

$$\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{dy}{dz} \left(-\frac{1}{x^2}\right) = \frac{dy}{dz} (-z^2)$$

$$\frac{d^2y}{dz^2} = \frac{d}{dx} \left(\frac{dy}{dz}\right) = \frac{dz}{dx} \frac{d}{dz} \left[\frac{dy}{dz} (-z^2)\right] = -\frac{1}{x^2} \left(-z^2 \frac{d^2y}{dz^2} - 2z \frac{dy}{dz}\right) = -z^2 \left(-z^2 \frac{d^2y}{dz^2} - 2z \frac{dy}{dz}\right)$$

As a result, the ODE in terms of $z$ is

$$\left(1 - \frac{1}{z^2}\right) \left[-z^2 \left(-z^2 \frac{d^2y}{dz^2} - 2z \frac{dy}{dz}\right)\right] - \frac{2}{z} \frac{dy}{dz} (-z^2) + l(l+1)y = 0,$$

or after simplifying,

$$(z^4 - z^2) \frac{d^2y}{dz^2} + 2z^3 \frac{dy}{dz} + l(l+1)y = 0.$$

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Divide both sides by \( z^4 - z^2 \) so that the coefficient of \( \frac{d^2 y}{dz^2} \) is 1.

\[
\frac{d^2 y}{dz^2} + \frac{2z^3}{z^4 - z^2} \frac{dy}{dz} + \frac{l(l+1)}{z^4 - z^2} y = 0
\]

At least one of the denominators is equal to zero at \( z = 0 \), so \( z = 0 \) is a singular point. Since the following limits are finite, it is in fact regular.

\[
\lim_{z \to 0} z \left( \frac{2z^3}{z^4 - z^2} \right) = \lim_{z \to 0} \frac{2z^2}{z^2 - 1} = 0
\]

\[
\lim_{z \to 0} \frac{l(l+1)}{z^4 - z^2} = \lim_{z \to 0} \frac{l(l+1)}{z^2 - 1} = -l(l+1)
\]

Therefore, \( x = \infty \) is a regular singular point of the Legendre equation.