Exercise 7.4.1

Show that Legendre's equation has regular singularities at $x = -1$, 1, and ∞.

Solution

Legendre's equation is a second-order linear homogeneous ODE.

$$(1 - x^2)y'' - 2xy' + l(l+1)y = 0$$

Divide both sides by $1 - x^2$ so that the coefficient of y'' is 1.

$$y'' - \frac{2x}{1-x^2}y' + \frac{l(l+1)}{1-x^2}y = 0$$

There are singular points where the denominators are equal to zero: $x = \pm 1$. $x = -1$ is regular because the following limits are finite.

$$\lim_{x\to -1}(x+1)\left(-\frac{2x}{1-x^2}\right) = \lim_{x\to -1}\left(-\frac{2x}{1-x}\right) = 1$$
$$\lim_{x\to -1}(x+1)^2\frac{l(l+1)}{1-x^2} = \lim_{x\to -1}\frac{l(l+1)(x+1)}{1-x} = 0$$

$x = 1$ is regular for the same reason.

$$\lim_{x\to 1}(x-1)\left(-\frac{2x}{1-x^2}\right) = \lim_{x\to 1}\left(\frac{2x}{1+x}\right) = 1$$
$$\lim_{x\to 1}(x-1)^2\frac{l(l+1)}{1-x^2} = \lim_{x\to 1}\frac{l(l+1)(1-x)}{1+x} = 0$$

In order to investigate the behavior at $x = \infty$, make the substitution,

$$x = \frac{1}{z},$$

in Legendre's equation.

$$(1 - x^2)y'' - 2xy' + l(l+1)y = 0 \quad \rightarrow \quad \left(1 - \frac{1}{z^2}\right)y'' - \frac{2}{z}y' + l(l+1)y = 0$$

Use the chain rule to find what the derivatives of y are in terms of this new variable.

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\left(-\frac{1}{x^2}\right) = \frac{dy}{dz}(-z^2)$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dz}{dx}\frac{d}{dz}\left[\frac{dy}{dz}(-z^2)\right] = -\frac{1}{x^2}\left(-z^2\frac{d^2y}{dz^2} - 2z\frac{dy}{dz}\right) = -z^2\left(-z^2\frac{d^2y}{dz^2} - 2z\frac{dy}{dz}\right)$$

As a result, the ODE in terms of z is

$$\left(1 - \frac{1}{z^2}\right)\left[-z^2\left(-z^2\frac{d^2y}{dz^2} - 2z\frac{dy}{dz}\right)\right] - \frac{2}{z}\frac{dy}{dz}(-z^2) + l(l+1)y = 0,$$

or after simplifying,

$$(z^4 - z^2)\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz} + l(l+1)y = 0.$$

Divide both sides by $z^4 - z^2$ so that the coefficient of d^2y/dz^2 is 1.

$$\frac{d^2y}{dz^2} + \frac{2z^3}{z^4 - z^2}\frac{dy}{dz} + \frac{l(l+1)}{z^4 - z^2}y = 0$$

At least one of the denominators is equal to zero at $z = 0$, so $z = 0$ is a singular point. Since the following limits are finite, it is in fact regular.

$$\lim_{z\to 0} z\left(\frac{2z^3}{z^4 - z^2}\right) = \lim_{z\to 0}\frac{2z^2}{z^2 - 1} = 0$$
$$\lim_{z\to 0} z^2\left[\frac{l(l+1)}{z^4 - z^2}\right] = \lim_{z\to 0}\frac{l(l+1)}{z^2 - 1} = -l(l+1)$$

Therefore, $x = \infty$ is a regular singular point of the Legendre equation.