Exercise 7.4.2

Show that Laguerre's equation, like the Bessel equation, has a regular singularity at $x = 0$ and an irregular singularity at $x = \infty$.

Solution

Laguerre's equation is a second-order linear homogeneous ODE.

$$xy'' + (1 - x)y' + ay = 0$$

Divide both sides by x so that the coefficient of y'' is 1.

$$y'' + \frac{1 - x}{x}y' + \frac{a}{x}y = 0$$

There are singular points where the denominators are equal to zero: $x = 0$. $x = 0$ is regular because the following limits are finite.

$$\lim_{x\to 0} x\left(\frac{1 - x}{x}\right) = \lim_{x\to 0}(1 - x) = 1$$
$$\lim_{x\to 0} x^2\left(\frac{a}{x}\right) = \lim_{x\to 0} ax = 0$$

In order to investigate the behavior at $x = \infty$, make the substitution,

$$x = \frac{1}{z},$$

in Laguerre's equation.

$$xy'' + (1 - x)y' + ay = 0 \quad \rightarrow \quad \frac{1}{z}y'' + \left(1 - \frac{1}{z}\right)y' + ay = 0$$

Use the chain rule to find what the derivatives of y are in terms of this new variable.

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\left(-\frac{1}{x^2}\right) = \frac{dy}{dz}(-z^2)$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dz}{dx}\frac{d}{dz}\left[\frac{dy}{dz}(-z^2)\right] = -\frac{1}{x^2}\left(-z^2\frac{d^2y}{dz^2} - 2z\frac{dy}{dz}\right) = -z^2\left(-z^2\frac{d^2y}{dz^2} - 2z\frac{dy}{dz}\right)$$

As a result, the ODE in terms of z is

$$\frac{1}{z}\left[-z^2\left(-z^2\frac{d^2y}{dz^2} - 2z\frac{dy}{dz}\right)\right] + \left(1 - \frac{1}{z}\right)\frac{dy}{dz}(-z^2) + ay = 0,$$

or after simplifying,

$$z^3\frac{d^2y}{dz^2} + (z^2 + z)\frac{dy}{dz} + ay = 0.$$

Divide both sides by z^3 so that the coefficient of d^2y/dz^2 is 1.

$$\frac{d^2y}{dz^2} + \frac{z^2 + z}{z^3}\frac{dy}{dz} + \frac{a}{z^3}y = 0$$

At least one of the denominators is equal to zero at $z = 0$, so $z = 0$ is a singular point. Since at least one of the following limits is infinite, it is in fact irregular.

$$\lim_{z\to 0} z\left(\frac{z^2+z}{z^3}\right) = \lim_{z\to 0}\left(1 - \frac{1}{z}\right) = \infty$$
$$\lim_{z\to 0} z^2\left(\frac{a}{z^3}\right) = \lim_{z\to 0}\frac{a}{z} = \infty$$

Therefore, $x = \infty$ is an irregular singular point of the Laguerre equation.