Exercise 7.4.2

Show that Laguerre's equation, like the Bessel equation, has a regular singularity at \( x = 0 \) and an irregular singularity at \( x = \infty \).

Solution

Laguerre's equation is a second-order linear homogeneous ODE.

\[
xy'' + (1 - x)y' + ay = 0
\]

Divide both sides by \( x \) so that the coefficient of \( y'' \) is 1.

\[
y'' + \frac{1 - x}{x}y' + \frac{a}{x}y = 0
\]

There are singular points where the denominators are equal to zero: \( x = 0 \). \( x = 0 \) is regular because the following limits are finite.

\[
\lim_{x \to 0} x \left( \frac{1 - x}{x} \right) = \lim_{x \to 0} (1 - x) = 1
\]

\[
\lim_{x \to 0} x^2 \left( \frac{a}{x} \right) = \lim_{x \to 0} ax = 0
\]

In order to investigate the behavior at \( x = \infty \), make the substitution,

\[
x = \frac{1}{z},
\]

in Laguerre's equation.

\[
xy'' + (1 - x)y' + ay = 0 \quad \rightarrow \quad \frac{1}{z}y'' + \left(1 - \frac{1}{z}\right)y' + ay = 0
\]

Use the chain rule to find what the derivatives of \( y \) are in terms of this new variable.

\[
\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{dy}{dz} \left( -\frac{1}{z^2} \right) = \frac{dy}{dz}(-z^2)
\]

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{dz}{dx} \frac{d}{dz} \left[ \frac{dy}{dz}(-z^2) \right] = -\frac{1}{x^2} \left( -z^2 \frac{d^2y}{dz^2} - 2z \frac{dy}{dz} \right) = -z^2 \left( -z^2 \frac{d^2y}{dz^2} - 2z \frac{dy}{dz} \right)
\]

As a result, the ODE in terms of \( z \) is

\[
\frac{1}{z} \left[ z^2 \left( -z^2 \frac{d^2y}{dz^2} - 2z \frac{dy}{dz} \right) \right] + \left(1 - \frac{1}{z}\right) \frac{dy}{dz}(-z^2) + ay = 0,
\]

or after simplifying,

\[
z^3 \frac{d^2y}{dz^2} + (z^2 + z \frac{dy}{dz}) + ay = 0.
\]

Divide both sides by \( z^3 \) so that the coefficient of \( d^2y/dz^2 \) is 1.

\[
z^2 + z \frac{dy}{dz} + \frac{a}{z^3}y = 0
\]
At least one of the denominators is equal to zero at $z = 0$, so $z = 0$ is a singular point. Since at least one of the following limits is infinite, it is in fact irregular.

$$\lim_{z \to 0} z \left( \frac{z^2 + z}{z^3} \right) = \lim_{z \to 0} \left( 1 - \frac{1}{z} \right) = \infty$$

$$\lim_{z \to 0} z^2 \left( \frac{a}{z^3} \right) = \lim_{z \to 0} \frac{a}{z} = \infty$$

Therefore, $x = \infty$ is an irregular singular point of the Laguerre equation.