Exercise 7.4.4

Show that Hermite’s equation has no singularity other than an irregular singularity at \( x = \infty \).

Solution

Hermite’s equation is a second-order linear homogeneous ODE.

\[
y'' - 2xy' + 2\alpha y = 0
\]

Since neither of the coefficients of \( y \) and \( y' \) blow up, there are no singular points for finite values of \( x \). In order to investigate the behavior at \( x = \infty \), make the substitution,

\[
x = \frac{1}{z},
\]

in Hermite’s equation.

\[
y'' - 2xy' + 2\alpha y = 0 \quad \rightarrow \quad y'' - \frac{2}{z}y' + 2\alpha y = 0
\]

Use the chain rule to find what the derivatives of \( y \) are in terms of this new variable.

\[
\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{dy}{dz} = \frac{dy}{dz} (-z^2)
\]

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{dz}{dx} \frac{dy}{dz} \left[ \frac{dy}{dz} (-z^2) \right] = \frac{1}{x^2} \left( -z^2 \frac{d^2y}{dz^2} - 2z \frac{dy}{dz} \right) = -z^2 \left( -z^2 \frac{d^2y}{dz^2} - 2z \frac{dy}{dz} \right)
\]

As a result, the ODE in terms of \( z \) is

\[
\left[ -z^2 \left( -z^2 \frac{d^2y}{dz^2} - 2z \frac{dy}{dz} \right) \right] - 2z \left[ \frac{dy}{dz} (-z^2) \right] + 2\alpha y = 0,
\]

or after simplifying,

\[
z^4 \frac{d^2y}{dz^2} + (2z^3 + 2z) \frac{dy}{dz} + 2\alpha y = 0.
\]

Divide both sides by \( z^4 \) so that the coefficient of \( d^2y/dz^2 \) is 1.

\[
\frac{d^2y}{dz^2} + \frac{2z^3 + 2z}{z^4} \frac{dy}{dz} + \frac{2\alpha}{z^4} y = 0
\]

At least one of the denominators is equal to zero at \( z = 0 \), so \( z = 0 \) is a singular point. Since at least one of the following limits is infinite, it is in fact irregular.

\[
\lim_{z \to 0} z \left( \frac{2z^3 + 2z}{z^4} \right) = \lim_{z \to 0} \left( 2 + \frac{2}{z^2} \right) = \infty
\]

\[
\lim_{z \to 0} z^2 \left( \frac{2\alpha}{z^4} \right) = \lim_{z \to 0} \frac{2\alpha}{z^2} = \infty
\]

Therefore, \( x = \infty \) is an irregular singular point of the Hermite equation.

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