Exercise 7.4.4

Show that Hermite's equation has no singularity other than an irregular singularity at $x = \infty$.

Solution

Hermite's equation is a second-order linear homogeneous ODE.

$$y'' - 2xy' + 2\alpha y = 0$$

Since neither of the coefficients of y and y' blow up, there are no singular points for finite values of x. In order to investigate the behavior at $x = \infty$, make the substitution,

$$x = \frac{1}{z},$$

in Hermite's equation.

$$y'' - 2xy' + 2\alpha y = 0 \quad \rightarrow \quad y'' - \frac{2}{z}y' + 2\alpha y = 0$$

Use the chain rule to find what the derivatives of y are in terms of this new variable.

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\left(-\frac{1}{x^2}\right) = \frac{dy}{dz}(-z^2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dz}{dx}\frac{d}{dz}\left[\frac{dy}{dz}(-z^2)\right] = -\frac{1}{x^2}\left(-z^2\frac{d^2y}{dz^2} - 2z\frac{dy}{dz}\right) = -z^2\left(-z^2\frac{d^2y}{dz^2} - 2z\frac{dy}{dz}\right)$$

As a result, the ODE in terms of z is

$$\left[-z^2\left(-z^2\frac{d^2y}{dz^2} - 2z\frac{dy}{dz}\right)\right] - \frac{2}{z}\left[\frac{dy}{dz}(-z^2)\right] + 2\alpha y = 0,$$

or after simplifying,

$$z^4\frac{d^2y}{dz^2} + (2z^3 + 2z)\frac{dy}{dz} + 2\alpha y = 0.$$

Divide both sides by z^4 so that the coefficient of d^2y/dz^2 is 1.

$$\frac{d^2y}{dz^2} + \frac{2z^3 + 2z}{z^4}\frac{dy}{dz} + \frac{2\alpha}{z^4}y = 0$$

At least one of the denominators is equal to zero at $z = 0$, so $z = 0$ is a singular point. Since at least one of the following limits is infinite, it is in fact irregular.

$$\lim_{z\to 0} z\left(\frac{2z^3 + 2z}{z^4}\right) = \lim_{z\to 0}\left(2 + \frac{2}{z^2}\right) = \infty$$

$$\lim_{z\to 0} z^2\left(\frac{2\alpha}{z^4}\right) = \lim_{z\to 0}\frac{2\alpha}{z^2} = \infty$$

Therefore, $x = \infty$ is an irregular singular point of the Hermite equation.