Exercise 7.7.2

Find the general solutions to the following inhomogeneous ODEs:

$$y'' + y = 1.$$

Solution

This is a linear ODE, so its general solution can be written as a sum of the complementary solution and the particular solution.

$$y(x) = y_c(x) + y_p(x)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + y_c = 0$$

Because it's homogeneous and the coefficients on the left side are constant, the solution for y_c is of the form e^{rx}.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Substitute these formulas into the ODE.

$$r^2 e^{rx} + e^{rx} = 0$$

Divide both sides by e^{rx}.

$$r^2 + 1 = 0$$

$$r = \{-i, i\}$$

Two solutions to the ODE are $y_c = e^{-ix}$ and $y_c = e^{ix}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} y_c(x) &= C_1 e^{-ix} + C_2 e^{ix} \\ &= C_1[\cos(-x) + i\sin(-x)] + C_2[\cos(x) + i\sin(x)] \\ &= C_1(\cos x - i\sin x) + C_2(\cos x + i\sin x) \\ &= (C_1 + C_2)\cos x + (-iC_1 + iC_2)\sin x \\ &= C_3\cos x + C_4\sin x \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p'' + y_p = 1. \tag{1}$$

Because the inhomogeneous term is a constant, y_p is expected to be a constant as well: $y_p(x) = A$. Substitute this formula into equation (1) to determine A.

$$(A)'' + (A) = 1 \quad \rightarrow \quad A = 1$$

Therefore, the particular solution is $y_p(x) = 1$, and the general solution to the original ODE is

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= C_3\cos x + C_4\sin x + 1. \end{aligned}$$