Exercise 7.7.5

Find the general solutions to the following inhomogeneous ODEs:

$$xy'' - (1 + x)y' + y = x^2.$$

Solution

This is a linear ODE, so its general solution can be written as a sum of the complementary solution and the particular solution.

$$y(x) = y_c(x) + y_p(x)$$

The complementary solution satisfies the associated homogeneous equation.

$$xy_c'' - (1 + x)y_c' + y_c = 0 \tag{1}$$

This ODE neither has constant coefficients nor is equidimensional, so it would seem a series solution is needed. Notice, though, that the coefficients add to zero. That means $y_c = e^x$ is a solution. Use the method of reduction of order to obtain the general solution: Plug in $y_c = C(x)e^x$ into equation (1) and solve the resulting ODE for $C(x)$.

$$x[C(x)e^x]'' - (1 + x)[C(x)e^x]' + [C(x)e^x] = 0$$

$$x[C'(x)e^x + C(x)e^x]' - (1 + x)[C'(x)e^x + C(x)e^x] + [C(x)e^x] = 0$$

$$x[C''(x)e^x + 2C'(x)e^x + C(x)e^x] - (1 + x)[C'(x)e^x + C(x)e^x] + [C(x)e^x] = 0$$

Simplify the left side.

$$xC''(x)e^x + xC'(x)e^x - C'(x)e^x = 0$$

Divide both sides by e^x and factor $C'(x)$.

$$xC''(x) + (x - 1)C'(x) = 0$$

Solve for $C''(x)/C'(x)$.

$$\frac{C''(x)}{C'(x)} = \frac{1 - x}{x}$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dx} \ln |C'(x)| = \frac{1 - x}{x}$$

The absolute value sign is included just because the argument of logarithm can't be negative. Integrate both sides with respect to x.

$$\begin{aligned} \ln |C'(x)| &= \int^x \frac{1 - s}{s} \, ds + C_1 \\ &= \ln x - x + C_1 \end{aligned}$$

Exponentiate both sides.

$$\begin{aligned} |C'(x)| &= e^{\ln x - x + C_1} \\ &= e^{\ln x} e^{-x} e^{C_1} \\ &= xe^{-x} e^{C_1} \end{aligned}$$

Remove the absolute value sign on the left by placing $\pm$ on the right.

$$C'(x) = \pm e^{C_1} x e^{-x}$$

Use a new constant C_2 for $\pm e^{C_1}$.

$$C'(x) = C_2 x e^{-x}$$

Integrate both sides with respect to x.

$$C(x) = -C_2(x+1)e^{-x} + C_3$$

Using a new constant C_4 for $-C_2$, we get

$$C(x) = C_4(x+1)e^{-x} + C_3,$$

so the complementary solution is

$$\begin{aligned} y_c(x) &= C(x)e^x \\ &= C_4(x+1) + C_3 e^x. \end{aligned}$$

On the other hand, the particular solution satisfies

$$xy_p'' - (1+x)y_p' + y_p = x^2. \tag{2}$$

Since the inhomogeneous term is a monomial of second power, the particular solution would be expected to be a linear combination of monomials up to and including the power of 2: $y_p(x) = A + Bx + Dx^2$. However, because 1 and x both satisfy the associated homogeneous equation, an extra factor of x^2 is needed. The trial solution is then $y_p(x) = x^2(A + Bx + Dx^2)$. Substitute this formula into equation (2) to determine A, B, and D.

$$x[x^2(A+Bx+Dx^2)]'' - (1+x)[x^2(A+Bx+Dx^2)]' + [x^2(A+Bx+Dx^2)] = x^2$$

$$x(2A+6Bx+12Dx^2) - (1+x)(2Ax+3Bx^2+4Dx^3) + [x^2(A+Bx+Dx^2)] = x^2$$

$$(2A-2A)x + (6B-3B-2A+A)x^2 + (12D-4D-3B+B)x^3 + (-4D+D)x^4 = x^2$$

Match the coefficients on both sides to obtain a system of equations for A, B, and D.

$$\begin{aligned} 2A - 2A &= 0 \\ 6B - 3B - 2A + A &= 1 \\ 12D - 4D - 3B + B &= 0 \\ -4D + D &= 0 \end{aligned}$$

Solving it yields $A = -1$, $B = 0$, and $D = 0$. Therefore, the particular solution is $y_p(x) = x^2(-1) = -x^2$, and the general solution to the original ODE is

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= C_4(x+1) + C_3 e^x - x^2. \end{aligned}$$