Exercise 7.7.5

Find the general solutions to the following inhomogeneous ODEs:

\[ xy'' - (1 + x)y' + y = x^2. \]

Solution

This is a linear ODE, so its general solution can be written as a sum of the complementary solution and the particular solution.

\[ y(x) = y_c(x) + y_p(x) \]

The complementary solution satisfies the associated homogeneous equation.

\[ xy'' - (1 + x)y' + y = 0 \] (1)

This ODE neither has constant coefficients nor is equidimensional, so it would seem a series solution is needed. Notice, though, that the coefficients add to zero. That means \( y_c = e^x \) is a solution. Use the method of reduction of order to obtain the general solution: Plug in \( y_c = C(x)e^x \) into equation (1) and solve the resulting ODE for \( C(x) \).

\[
x[C(x)e^x]' - (1 + x)[C(x)e^x]' + [C(x)e^x] = 0
\]

\[
x[C'(x)e^x + C(x)e^x] - (1 + x)[C'(x)e^x + C(x)e^x] + [C(x)e^x] = 0
\]

\[
x[C''(x)e^x + 2C'(x)e^x + C(x)e^x] - (1 + x)[C''(x)e^x + C(x)e^x] + [C(x)e^x] = 0
\]

Simplify the left side.

\[
xC''(x)e^x + xC'(x)e^x - C'(x)e^x = 0
\]

Divide both sides by \( e^x \) and factor \( C''(x) \).

\[
xC''(x) + (x - 1)C'(x) = 0
\]

Solve for \( C''(x)/C'(x) \).

\[
\frac{C''(x)}{C'(x)} = \frac{1 - x}{x}
\]

The left side can be written as the derivative of a logarithm by the chain rule.

\[
\frac{d}{dx} \ln |C'(x)| = \frac{1 - x}{x}
\]

The absolute value sign is included just because the argument of logarithm can’t be negative. Integrate both sides with respect to \( x \).

\[
\ln |C'(x)| = \int^x \frac{1 - s}{s} ds + C_1
\]

\[
= \ln x - x + C_1
\]

Exponentiate both sides.

\[
|C'(x)| = e^{\ln x - x + C_1}
\]

\[
= e^{\ln x} e^{-x} e^{C_1}
\]

\[
= xe^{-x} e^{C_1}
\]

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Remove the absolute value sign on the left by placing $\pm$ on the right.

$$C'(x) = \pm e^{C_1}xe^{-x}$$

Use a new constant $C_2$ for $\pm e^{C_1}$.

$$C'(x) = C_2xe^{-x}$$

Integrate both sides with respect to $x$.

$$C(x) = -C_2(x + 1)e^{-x} + C_3$$

Using a new constant $C_4$ for $-C_2$, we get

$$C(x) = C_4(x + 1)e^{-x} + C_3,$$

so the complementary solution is

$$y_c(x) = e^{C(x)}e^x = C_4(x + 1) + C_3e^x.$$ 

On the other hand, the particular solution satisfies

$$xy'' - (1 + x)y' + yp = x^2. \quad (2)$$

Since the inhomogeneous term is a monomial of second power, the particular solution would be expected to be a linear combination of monomials up to and including the power of 2:

$$y_p(x) = A + Bx + Dx^2.$$ 

However, because 1 and $x$ both satisfy the associated homogeneous equation, an extra factor of $x^2$ is needed. The trial solution is then $y_p(x) = x^2(A + Bx + Dx^2)$.

Substitute this formula into equation (2) to determine $A$, $B$, and $D$.

$$x[x^2(A + Bx + Dx^2)]'' - (1 + x)[x^2(A + Bx + Dx^2)]' + [x^2(A + Bx + Dx^2)] = x^2$$

$$x(2A + 6Bx + 12Dx^2) - (1 + x)(2Ax + 3Bx^2 + 4Dx^3) + [x^2(A + Bx + Dx^2)] = x^2$$

$$x^2(2A - 2A)x + (6B - 3B - 2A + A)x^2 + (12D - 4D - 3B + B)x^3 + (-4D + D)x^4 = x^2$$

Match the coefficients on both sides to obtain a system of equations for $A$, $B$, and $D$.

$$2A - 2A = 0$$

$$6B - 3B - 2A + A = 1$$

$$12D - 4D - 3B + B = 0$$

$$-4D + D = 0$$

Solving it yields $A = -1$, $B = 0$, and $D = 0$. Therefore, the particular solution is

$$y_p(x) = x^2(-1) = -x^2,$$ 

and the general solution to the original ODE is

$$y(x) = y_c(x) + y_p(x)$$

$$= C_4(x + 1) + C_3e^x - x^2.$$