

Exercise 8.2.1

Show that Laguerre's ODE, Table 7.1, may be put into self-adjoint form by multiplying by e^{-x} and that $w(x) = e^{-x}$ is the weighting function.

Solution

From Table 7.1 on page 345, Laguerre's equation is

$$xy'' + (1 - x)y' + ay = 0.$$

At the moment it is not self-adjoint because

$$\frac{d}{dx}(x) \neq 1 - x.$$

However, if both sides of Laguerre's equation are multiplied by the weight function $w(x) = e^{-x}$, then it becomes self-adjoint

$$xe^{-x}y'' + (1 - x)e^{-x}y' + ae^{-x}y = 0 \tag{1}$$

because

$$\begin{aligned} \frac{d}{dx}(xe^{-x}) &= e^{-x} - xe^{-x} \\ &= (1 - x)e^{-x}. \end{aligned}$$

Equation (1) can therefore be written as

$$xe^{-x}y'' + \frac{d}{dx}(xe^{-x})y' + ae^{-x}y = 0,$$

or

$$\frac{d}{dx}(xe^{-x}y') + ae^{-x}y = 0.$$