Exercise 9.2.1

Find the general solutions of the PDEs in Exercises 9.2.1 to 9.2.4.
\[
\frac{\partial \psi}{\partial x} + 2 \frac{\partial \psi}{\partial y} + (2x - y)\psi = 0.
\]

Solution

Since \( \psi \) is a function of two variables \( \psi = \psi(x, y) \), its differential is defined as
\[
d\psi = \frac{\partial \psi}{\partial x} \, dx + \frac{\partial \psi}{\partial y} \, dy.
\]
Dividing both sides by \( dx \), we obtain the relationship between the total derivative of \( \psi \) and the partial derivatives of \( \psi \).
\[
\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{dy}{dx} \frac{\partial \psi}{\partial y}
\]
In light of this, the PDE reduces to the ODE,
\[
\frac{d\psi}{dx} + (2x - y)\psi = 0, \quad (1)
\]
along the characteristic curves in the \( xy \)-plane that satisfy
\[
\frac{dy}{dx} = 2, \quad y(0, \xi) = \xi, \quad (2)
\]
where \( \xi \) is a characteristic coordinate. Integrate both sides of equation (2) with respect to \( x \) to solve for \( y(x, \xi) \).
\[
y(x, \xi) = 2x + \xi
\]
From this equation we see that \( 2x - y = -\xi \), which means equation (1) becomes
\[
\frac{d\psi}{dx} - \xi \psi = 0.
\]
Solve this ODE by separating variables.
\[
\frac{d\psi}{\psi} = \xi \, dx
\]
Integrate both sides.
\[
\int \frac{d\psi}{\psi} = \int \xi \, dx
\]
\[
\ln |\psi| = \xi x + f(\xi)
\]
Here \( f \) is an arbitrary function of the characteristic coordinate \( \xi \). Exponentiate both sides.
\[
|\psi| = e^{\xi x + f(\xi)} = e^{f(\xi)} e^{\xi x}
\]
Introduce \( \pm \) on the right side to remove the absolute value sign.
\[
\psi(x, \xi) = \pm e^{f(\xi)} e^{\xi x}
\]
Use a new arbitrary function \( g(\xi) \) for \( \pm e^{f(\xi)} \).
\[
\psi(x, \xi) = g(\xi) e^{\xi x}
\]
Therefore, since \( \xi = y - 2x \),
\[
\psi(x, y) = g(y - 2x) e^{x(y - 2x)}.
\]

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