Exercise 9.2.5

(a) Show that the PDE

$$y\frac{\partial\psi}{\partial x} + x\frac{\partial\psi}{\partial y} = 0$$

can be transformed into a readily soluble form by writing it in the new variables $u = xy$, $v = x^2 - y^2$, and find its general solution.

(b) Discuss this result in terms of characteristics.

Solution

Part (a)

Make the change of variables,

$$u = xy \qquad v = x^2 - y^2.$$

The aim now is to find $\partial\psi/\partial x$ and $\partial\psi/\partial y$ in terms of these new variables. Use the chain rule.

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\psi}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial\psi}{\partial u}(y) + \frac{\partial\psi}{\partial v}(2x) = y\frac{\partial\psi}{\partial u} + 2x\frac{\partial\psi}{\partial v}$$
$$\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\psi}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial\psi}{\partial u}(x) + \frac{\partial\psi}{\partial v}(-2y) = x\frac{\partial\psi}{\partial u} - 2y\frac{\partial\psi}{\partial v}$$

Consequently, the transformed PDE is

$$y\left(y\frac{\partial\psi}{\partial u} + 2x\frac{\partial\psi}{\partial v}\right) + x\left(x\frac{\partial\psi}{\partial u} - 2y\frac{\partial\psi}{\partial v}\right) = 0$$

$$y^2\frac{\partial\psi}{\partial u} + \cancel{2xy\frac{\partial\psi}{\partial v}} + x^2\frac{\partial\psi}{\partial u} - \cancel{2xy\frac{\partial\psi}{\partial v}} = 0$$

$$(y^2 + x^2)\frac{\partial\psi}{\partial u} = 0.$$

Divide both sides by $x^2 + y^2$.

$$\frac{\partial\psi}{\partial u} = 0.$$

This equation indicates that ψ has no dependence on u, so

$$\psi(u, v) = f(v),$$

where f is an arbitrary function. Now eliminate u and v in favor of x and y.

$$\psi(x, y) = f(x^2 - y^2)$$

Part (b)

The solution to the PDE is the same all along each hyperbola in the xy-plane.

$$x^2 - y^2 = \xi$$

These are the characteristic curves.