Exercise 9.2.2

Find the general solutions of the PDEs in Exercises 9.2.1 to 9.2.4.

$$\frac{\partial\psi}{\partial x} - 2\frac{\partial\psi}{\partial y} + x + y = 0.$$

Solution

Since ψ is a function of two variables $\psi = \psi(x, y)$, its differential is defined as

$$d\psi = \frac{\partial\psi}{\partial x}\,dx + \frac{\partial\psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the relationship between the total derivative of ψ and the partial derivatives of ψ.

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{dy}{dx}\frac{\partial\psi}{\partial y}$$

In light of this, the PDE reduces to the ODE,

$$\frac{d\psi}{dx} + x + y = 0, \tag{1}$$

along the characteristic curves in the xy-plane that satisfy

$$\frac{dy}{dx} = -2, \qquad y(0, \xi) = \xi, \tag{2}$$

where ξ is a characteristic coordinate. Integrate both sides of equation (2) with respect to x to solve for $y(x, \xi)$.

$$y(x, \xi) = -2x + \xi$$

As a result, equation (1) becomes

$$\frac{d\psi}{dx} + x + (-2x + \xi) = 0 \quad \rightarrow \quad \frac{d\psi}{dx} = x - \xi.$$

Solve this ODE by integrating both sides with respect to x.

$$\psi(x, \xi) = \frac{x^2}{2} - \xi x + f(\xi)$$

Here f is an arbitrary function of the characteristic coordinate ξ. Now eliminate ξ in favor of x and y: $\xi = 2x + y$.

$$\begin{aligned}\psi(x, y) &= \frac{x^2}{2} - x(2x + y) + f(2x + y)\\ &= \frac{x^2}{2} - 2x^2 - xy + f(2x + y)\end{aligned}$$

Therefore,

$$\psi(x, y) = -\frac{3}{2}x^2 - xy + f(2x + y).$$