Exercise 9.2.4

Find the general solutions of the PDEs in Exercises 9.2.1 to 9.2.4.

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial z} = x - y.$$

Solution

Since ψ is a function of three variables $\psi = \psi(x, y, z)$, its differential is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy + \frac{\partial \psi}{\partial z}\, dz.$$

Dividing both sides by dx, we obtain the relationship between the total derivative of ψ and the partial derivatives of ψ.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{dy}{dx}\frac{\partial \psi}{\partial y} + \frac{dz}{dx}\frac{\partial \psi}{\partial z}$$

In light of this, the PDE reduces to the ODE,

$$\frac{d\psi}{dx} = x - y, \tag{1}$$

along the characteristic curves that satisfy

$$\frac{dy}{dx} = 1, \qquad y(0, \xi) = \xi, \tag{2}$$

$$\frac{dz}{dx} = 1, \qquad z(0, \eta) = \eta, \tag{3}$$

where ξ and η are characteristic coordinates. Integrate both sides of equations (2) and (3) with respect to x.

$$y(x, \xi) = x + \xi$$
$$z(x, \eta) = x + \eta$$

From the first of these equations, we see that $x - y = -\xi$, so equation (1) becomes

$$\frac{d\psi}{dx} = -\xi.$$

Integrate both sides with respect to x.

$$\psi(x, \xi, \eta) = -\xi x + f(\xi, \eta)$$

Here f is an arbitrary function of the two characteristic coordinates. Now eliminate ξ and η in favor of x, y, and z: $\xi = y - x$ and $\eta = z - x$.

$$\psi(x, y, z) = -(y - x)x + f(y - x, z - x)$$

Therefore,

$$\psi(x, y, z) = x(x - y) + f(y - x, z - x).$$