Exercise 9.4.6

The quantum mechanical angular momentum operator is given by $\mathbf{L} = -i(\mathbf{r} \times \nabla)$. Show that

$$\mathbf{L} \cdot \mathbf{L}\psi = l(l+1)\psi$$

leads to the associated Legendre equation.
Hint. Section 8.3 and Exercise 8.3.1 may be helpful.

Solution

Expand $\mathbf{r}$ and ∇ in spherical polar coordinates (r, θ, φ), where θ is the angle from the polar axis.

$$\begin{aligned}
\mathbf{L} \cdot \mathbf{L} &= [-i(\mathbf{r} \times \nabla)] \cdot [-i(\mathbf{r} \times \nabla)] \\
&= (-i)^2(\mathbf{r} \times \nabla) \cdot (\mathbf{r} \times \nabla) \\
&= -(\mathbf{r} \times \nabla) \cdot (\mathbf{r} \times \nabla) \\
&= -\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\varphi}} \\ r & 0 & 0 \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \end{vmatrix} \cdot \begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\varphi}} \\ r & 0 & 0 \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \end{vmatrix} \\
&= -\left(-\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial}{\partial\varphi} + \hat{\boldsymbol{\varphi}}\frac{\partial}{\partial\theta}\right) \cdot \left(-\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial}{\partial\varphi} + \hat{\boldsymbol{\varphi}}\frac{\partial}{\partial\theta}\right) \\
&= -\left[\left(-\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \cdot \left(-\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial}{\partial\varphi} + \hat{\boldsymbol{\varphi}}\frac{\partial}{\partial\theta}\right) + \left(\hat{\boldsymbol{\varphi}}\frac{\partial}{\partial\theta}\right) \cdot \left(-\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial}{\partial\varphi} + \hat{\boldsymbol{\varphi}}\frac{\partial}{\partial\theta}\right)\right] \\
&= -\left[\left(-\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\right) \cdot \left(-\frac{\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\varphi}}{\sin\theta}\frac{\partial}{\partial\varphi} - \frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial\hat{\boldsymbol{\varphi}}}{\partial\varphi}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\varphi}}\frac{\partial^2}{\partial\theta\partial\varphi}\right)\right. \\
&\qquad \left. + \hat{\boldsymbol{\varphi}} \cdot \left(-\frac{\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta}}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{\hat{\boldsymbol{\theta}}}{\sin^2\theta}(\cos\theta)\frac{\partial}{\partial\varphi} - \frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi} + \frac{\partial\hat{\boldsymbol{\varphi}}}{\partial\theta}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\varphi}}\frac{\partial^2}{\partial\theta^2}\right)\right] \\
&= -\left[\left(-\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\right) \cdot \left(-\frac{\hat{\boldsymbol{\varphi}}\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi} - \frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial^2}{\partial\varphi^2} + (-\hat{\mathbf{r}}\sin\theta - \hat{\boldsymbol{\theta}}\cos\theta)\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\varphi}}\frac{\partial^2}{\partial\theta\partial\varphi}\right)\right. \\
&\qquad \left. + \hat{\boldsymbol{\varphi}} \cdot \left(-\frac{(-\hat{\mathbf{r}})}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{\hat{\boldsymbol{\theta}}}{\sin^2\theta}(\cos\theta)\frac{\partial}{\partial\varphi} - \frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi} + (0)\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\varphi}}\frac{\partial^2}{\partial\theta^2}\right)\right] \\
&= -\left[\left(-\frac{1}{\sin\theta}\right)\left(-\frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2} - \cos\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\theta^2}\right] \\
&= -\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}\right) \\
&= -\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]
\end{aligned}$$

Consequently, the governing PDE for ψ is

$$-\left[\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)\right] = l(l+1)\psi,$$

or

$$\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+l(l+1)\psi=0.$$

It is linear and homogeneous, so the method of separation of variables can be applied to solve it. Assume a product solution of the form $\psi(\theta,\varphi)=\Theta(\theta)\Phi(\varphi)$ and plug it into the PDE.

$$\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}[\Theta(\theta)\Phi(\varphi)]+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}[\Theta(\theta)\Phi(\varphi)]\right]+l(l+1)[\Theta(\theta)\Phi(\varphi)]=0$$

$$\frac{\Theta(\theta)}{\sin^2\theta}\Phi''(\varphi)+\frac{\Phi(\varphi)}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+l(l+1)\Theta(\theta)\Phi(\varphi)=0$$

Divide both sides by $\Theta(\theta)\Phi(\varphi)$.

$$\frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi}+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+l(l+1)=0$$

Multiply both sides by $\sin^2\theta$ and bring the first term to the right side.

$$\underbrace{\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+l(l+1)\sin^2\theta}_{\text{function of }\theta}=\underbrace{-\frac{\Phi''}{\Phi}}_{\text{function of }\varphi}$$

The only way a function of θ is equal to a function of φ is if both are equal to a constant μ.

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+l(l+1)\sin^2\theta=-\frac{\Phi''}{\Phi}=\mu$$

By using the method of separation of variables, the PDE has reduced to two ODEs—one in θ and one in φ.

$$\left.\begin{aligned}\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+l(l+1)\sin^2\theta&=\mu\\-\frac{\Phi''}{\Phi}&=\mu\end{aligned}\right\}$$

In the ODE for θ, bring μ to the left side and multiply both sides by $\Theta/\sin^2\theta$.

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{\mu}{\sin^2\theta}\Theta+l(l+1)\Theta=0$$

Make the change of variables $t=\cos\theta$. Use the chain rule to write $d/d\theta$ in terms of this new variable.

$$\frac{d}{d\theta}=\frac{dt}{d\theta}\frac{d}{dt}=-\sin\theta\frac{d}{dt}\quad\rightarrow\quad\begin{cases}\dfrac{1}{\sin\theta}\dfrac{d}{d\theta}=-\dfrac{d}{dt}\\\sin\theta\dfrac{d}{d\theta}=-\sin^2\theta\dfrac{d}{dt}=-(1-\cos^2\theta)\dfrac{d}{dt}=-(1-t^2)\dfrac{d}{dt}\\\dfrac{1}{\sin^2\theta}=\dfrac{1}{1-\cos^2\theta}=\dfrac{1}{1-t^2}\end{cases}$$

The transformed ODE is the associated Legendre equation (with $\mu=m^2$).

$$-\frac{d}{dt}\left[-(1-t^2)\frac{d\Theta}{dt}\right]-\frac{\mu}{1-t^2}\Theta+l(l+1)\Theta=0\quad\rightarrow\quad(1-t^2)\Theta''(t)-2t\Theta'(t)-\frac{\mu}{1-t^2}\Theta(t)+l(l+1)\Theta(t)=0$$