Exercise 9.4.7

The 1-D Schrödinger wave equation for a particle in a potential field \( V = \frac{1}{2}kx^2 \) is

\[
- \frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{1}{2}kx^2 \psi = E\psi(x).
\]

(a) Defining

\[
a = \left( \frac{mk}{\hbar^2} \right)^{1/4}, \quad \lambda = \frac{2E}{\hbar} \left( \frac{m}{k} \right)^{1/2},
\]

and setting \( \xi = ax \), show that

\[
\frac{d^2 \psi(\xi)}{d\xi^2} + (\lambda - \xi^2)\psi(\xi) = 0.
\]

(b) Substituting

\[
\psi(\xi) = y(\xi)e^{-\xi^2/2},
\]

show that \( y(\xi) \) satisfies the Hermite differential equation.

Solution

Part (a)

Use the chain rule to write \( d/dx \) and \( d^2/dx^2 \) in terms of this new variable \( \xi \).

\[
\frac{d}{dx} = \frac{d\xi}{dx} \frac{d}{d\xi} = a \frac{d}{d\xi}
\]

\[
\frac{d^2}{dx^2} = \frac{d}{dx} \left( \frac{d}{dx} \right) = a \frac{d}{d\xi} \left( \frac{d}{d\xi} \right) = a^2 \frac{d^2}{d\xi^2}
\]

Consequently, the Schrödinger equation becomes

\[
- \frac{\hbar^2}{2m} \left( a^2 \frac{d^2}{d\xi^2} \right) \psi + \frac{1}{2}k \left( \frac{\xi}{a} \right)^2 \psi = E\psi
\]

\[
- \frac{\hbar^2}{2m} a^2 \frac{d^2 \psi}{d\xi^2} + \frac{k}{2}a^2 \xi^2 \psi = E\psi
\]

\[
- \frac{\hbar^2}{2m} \sqrt{mk} \frac{d^2 \psi}{d\xi^2} + \frac{h}{2} \sqrt{mk} \xi^2 \psi = E\psi
\]

\[
\frac{h}{2} \sqrt{\frac{k}{m}} \frac{d^2 \psi}{d\xi^2} + \frac{h}{2} \sqrt{\frac{k}{m}} \xi^2 \psi = E\psi.
\]

Multiply both sides by \(-\frac{2}{\hbar} \sqrt{\frac{m}{k}}\).

\[
\frac{d^2 \psi}{d\xi^2} - \xi^2 \psi = -\frac{2E}{\hbar} \sqrt{\frac{m}{k}} \psi
\]

Therefore,

\[
\frac{d^2 \psi(\xi)}{d\xi^2} + (\lambda - \xi^2)\psi(\xi) = 0.
\]
Part (b)

Make the substitution $\psi(\xi) = y(\xi)e^{-\xi^2/2}$ in the equation.

$$\frac{d^2}{d\xi^2}[y(\xi)e^{-\xi^2/2}] + (\lambda - \xi^2)[y(\xi)e^{-\xi^2/2}] = 0$$

$$\frac{d}{d\xi} \left( e^{-\xi^2/2}y' - \xi e^{-\xi^2/2}y \right) + (\lambda - \xi^2)e^{-\xi^2/2}y = 0$$

$$e^{-\xi^2/2}y'' - \xi e^{-\xi^2/2}y' - e^{-\xi^2/2}y - \xi(-\xi)e^{-\xi^2/2}y - \xi e^{-\xi^2/2}y + (\lambda - \xi^2)e^{-\xi^2/2}y = 0$$

$$e^{-\xi^2/2}y'' - 2\xi e^{-\xi^2/2}y' - e^{-\xi^2/2}y + \xi^2 e^{-\xi^2/2}y + \lambda e^{-\xi^2/2}y - \xi^2 e^{-\xi^2/2}y = 0$$

$$e^{-\xi^2/2}y'' - 2\xi e^{-\xi^2/2}y' + e^{-\xi^2/2}(\lambda - 1)y = 0$$

Multiply both sides by $e^{\xi^2/2}$ to obtain Hermite’s differential equation.

$$y'' - 2\xi y' + (\lambda - 1)y = 0$$