Exercise 9.5.1

Verify that the following are solutions of Laplace’s equation:

(a) \( \psi_1 = \frac{1}{r}, \ r \neq 0 \)  
(b) \( \psi_2 = \frac{1}{2r} \ln \frac{r + z}{r - z} \).

Solution

The Laplace equation is the following PDE.

\[ \nabla^2 \psi = 0 \]

Expand the Laplacian operator in spherical polar coordinates \((r, \theta, \varphi)\), where \(\theta\) is the angle from the polar axis.

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \varphi^2} = 0
\]

Part (a)

Substitute \(\psi = \frac{1}{r}\) into the equation.

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \left( \frac{1}{r} \right)}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \left( \frac{1}{r} \right)}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \left( \frac{1}{r} \right)}{\partial \varphi^2} = 0
\]

Because \(\psi\) is only dependent on \(r\), the angular derivatives vanish.

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \left( -\frac{1}{r^2} \right) \right) = 0
\]

\[
\frac{1}{r^2} \frac{\partial}{\partial r} (-1) = 0
\]

Therefore, \(\psi_1 = \frac{1}{r}\) is a solution to the Laplace equation.

Part (b)

Use \(r \cos \theta\) for \(z\) and substitute \(\psi = \left(\frac{1}{2r}\right) \ln [(r + z)/(r - z)]\) into the equation.

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \left( \frac{1}{2r} \ln \frac{1 + \cos \theta}{1 - \cos \theta} \right)}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \left( \frac{1}{2r} \ln \frac{1 + \cos \theta}{1 - \cos \theta} \right)}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \left( \frac{1}{2r} \ln \frac{1 + \cos \theta}{1 - \cos \theta} \right)}{\partial \varphi^2} = 0
\]

The third term is zero because \(\psi\) doesn’t depend on \(\varphi\).

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \left( -\frac{1}{2r^2} \right) \right) \ln \frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1}{2r^3 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{1 - \cos \theta}{1 + \cos \theta} \frac{\partial \left( \frac{1 + \cos \theta}{1 - \cos \theta} \right)}{\partial \theta} \right) = 0
\]

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\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( -\frac{1}{2} \right) \ln \frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1}{2r^3 \sin \theta} \frac{\partial}{\partial \theta} \left[ \sin \theta \frac{1 - \cos \theta}{1 + \cos \theta} \left( -\frac{2 \sin \theta}{(1 - \cos \theta)^2} \right) \right] = 0
\]

\[
\frac{1}{2r^3 \sin \theta} \frac{\partial}{\partial \theta} \left( -\frac{2 \sin^2 \theta}{1 - \cos^2 \theta} \right) = 0
\]

\[
-\frac{1}{r^3 \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{\sin^2 \theta}{1 - \cos^2 \theta} \right) = 0
\]

\[
-\frac{1}{r^3 \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{\sin^2 \theta}{\sin^2 \theta} \right) = 0
\]

\[
-\frac{1}{r^3 \sin \theta} \frac{\partial}{\partial \theta} (1) = 0
\]

\[
0 = 0
\]

Therefore, \( \psi_2 = \frac{1}{1/r^2} \ln[(r + z)/(r - z)] \) is a solution to the Laplace equation.