Exercise 9.5.1

Verify that the following are solutions of Laplace's equation:

$$\text{(a)} \quad \psi_1 = 1/r, \; r \neq 0, \quad \text{(b)} \quad \psi_2 = \frac{1}{2r} \ln \frac{r+z}{r-z}.$$

Solution

The Laplace equation is the following PDE.

$$\nabla^2 \psi = 0$$

Expand the Laplacian operator in spherical polar coordinates (r, θ, φ), where θ is the angle from the polar axis.

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \varphi^2} = 0$$

Part (a)

Substitute $\psi = 1/r$ into the equation.

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial}{\partial r} \left(\frac{1}{r} \right) \right] + \underbrace{\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left[\sin\theta \frac{\partial}{\partial \theta} \left(\frac{1}{r} \right) \right]}_{=0} + \underbrace{\frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \left(\frac{1}{r} \right)}_{=0} \stackrel{?}{=} 0$$

Because ψ is only dependent on r, the angular derivatives vanish.

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \left(-\frac{1}{r^2} \right) \right] \stackrel{?}{=} 0$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} (-1) \stackrel{?}{=} 0$$

$$0 = 0$$

Therefore, $\psi_1 = 1/r$ is a solution to the Laplace equation.

Part (b)

Use $r\cos\theta$ for z and substitute $\psi = (1/2r) \ln[(r+z)/(r-z)]$ into the equation.

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial}{\partial r} \left(\frac{1}{2r} \ln \frac{1+\cos\theta}{1-\cos\theta} \right) \right] + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left[\sin\theta \frac{\partial}{\partial \theta} \left(\frac{1}{2r} \ln \frac{1+\cos\theta}{1-\cos\theta} \right) \right]$$
$$+ \underbrace{\frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \left(\frac{1}{2r} \ln \frac{1+\cos\theta}{1-\cos\theta} \right)}_{=0} \stackrel{?}{=} 0$$

The third term is zero because ψ doesn't depend on φ.

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \left(-\frac{1}{2r^2} \right) \right] \ln \frac{1+\cos\theta}{1-\cos\theta} + \frac{1}{2r^3 \sin\theta} \frac{\partial}{\partial \theta} \left[\sin\theta \frac{1-\cos\theta}{1+\cos\theta} \frac{\partial}{\partial \theta} \left(\frac{1+\cos\theta}{1-\cos\theta} \right) \right] \stackrel{?}{=} 0$$

$$\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(-\frac{1}{2}\right)\ln\frac{1+\cos\theta}{1-\cos\theta}}_{=0}+\frac{1}{2r^3\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{1-\cos\theta}{1+\cos\theta}\left(-\frac{2\sin\theta}{(1-\cos\theta)^2}\right)\right]\stackrel{?}{=}0$$

$$\frac{1}{2r^3\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{2\sin^2\theta}{1-\cos^2\theta}\right)\stackrel{?}{=}0$$

$$-\frac{1}{r^3\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{1-\cos^2\theta}\right)\stackrel{?}{=}0$$

$$-\frac{1}{r^3\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{\sin^2\theta}\right)\stackrel{?}{=}0$$

$$-\frac{1}{r^3\sin\theta}\frac{\partial}{\partial\theta}(1)\stackrel{?}{=}0$$

$$0=0$$

Therefore, $\psi_2 = (1/2r)\ln[(r+z)/(r-z)]$ is a solution to the Laplace equation.