Exercise 9.5.2

If $\Psi$ is a solution of Laplace’s equation, $\nabla^2 \Psi = 0$, show that $\partial \Psi / \partial z$ is also a solution.

Solution

Suppose $\Psi$ is a solution of Laplace’s equation.

$$\nabla^2 \Psi = 0$$

Expand the Laplacian operator in Cartesian coordinates.

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = 0$$

Differentiate both sides with respect to $z$.

$$\frac{\partial}{\partial z} \left( \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} \right) = 0$$

The mixed derivatives are equal and can be arranged however we like.

$$\frac{\partial}{\partial x^2} \left( \frac{\partial \Psi}{\partial z} \right) + \frac{\partial}{\partial y^2} \left( \frac{\partial \Psi}{\partial z} \right) + \frac{\partial}{\partial z^2} \left( \frac{\partial \Psi}{\partial z} \right) = 0$$

Factor $\partial \Psi / \partial z$.

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left( \frac{\partial \Psi}{\partial z} \right) = 0$$

$$\nabla^2 \left( \frac{\partial \Psi}{\partial z} \right) = 0$$

$\partial \Psi / \partial z$ also satisfies Laplace’s equation, so it’s also a solution.