Exercise 9.5.3

Show that an argument based on Eq. (9.88) can be used to prove that the Laplace and Poisson equations with Dirichlet boundary conditions have unique solutions.

Solution

Eq. (9.88) in the text is Green's first identity with u and v both set equal to ψ.

$$\int_S \psi \frac{\partial \psi}{\partial \mathbf{n}}\, dS = \int_V \psi \nabla^2 \psi \, d\tau + \int_V \nabla\psi \cdot \nabla\psi \, d\tau \tag{9.88}$$

Suppose there are two solutions to the Poisson equation valid in some region V that satisfy a Dirichlet boundary condition on S, the boundary of V. Let these solutions be ψ_1 and ψ_2.

$$\begin{aligned} \nabla^2\psi_1 &= f, \quad \text{in } V & \nabla^2\psi_2 &= f, \quad \text{in } V \\ \psi_1 &= g, \quad \text{on } S & \psi_2 &= g, \quad \text{on } S \end{aligned}$$

Subtract both sides of the second PDE from those of the first. Do the same with the boundary conditions.

$$\begin{aligned} \nabla^2\psi_1 - \nabla^2\psi_2 &= f - f, \quad \text{in } V \\ \psi_1 - \psi_2 &= g - g, \quad \text{on } S \end{aligned}$$

As a result,

$$\begin{aligned} \nabla^2(\psi_1 - \psi_2) &= 0, \quad \text{in } V \\ \psi_1 - \psi_2 &= 0, \quad \text{on } S. \end{aligned}$$

Let $\psi = \psi_1 - \psi_2$.

$$\begin{aligned} \nabla^2\psi &= 0, \quad \text{in } V \\ \psi &= 0, \quad \text{on } S \end{aligned}$$

Now apply Eq. (9.88).

$$\int_S \overbrace{\psi}^{=0} \frac{\partial \psi}{\partial \mathbf{n}}\, dS = \int_V \psi \overbrace{\nabla^2 \psi}^{=0} \, d\tau + \int_V \nabla\psi \cdot \nabla\psi \, d\tau \tag{9.88}$$

All that remains is

$$\int_V |\nabla\psi|^2 \, d\tau = 0.$$

The integrand is zero according to the vanishing theorem.

$$\begin{aligned} |\nabla\psi|^2 &= 0 \\ \nabla\psi &= 0 \\ \psi &= \text{constant}, \quad \text{in } V \end{aligned}$$

In order for this solution to be consistent with the boundary condition, $\psi = 0$ on S, this constant must be zero.

$$\psi = 0, \quad \text{in } V$$

This means that the two solutions to the Poisson equation are one and the same function.

$$\psi_1 - \psi_2 = 0 \quad \rightarrow \quad \psi_1 = \psi_2$$

Therefore, the solution to the Poisson equation subject to a Dirichlet boundary condition has a unique solution. The same is true for the Laplace equation ($f = 0$).