Exercise 9.6.3

Solve the wave equation, Eq. (9.89), subject to the indicated conditions.

Determine $\psi(x,t)$ given that at $t = 0$ $\psi_0(x)$ is a single square-wave pulse as defined below, and the initial time derivative of $\psi$ is zero.

$$\psi_0(x) = 0, \ |x| > a/2, \ \psi_0(x) = 1/a, \ |x| < a/2.$$  

Solution

The initial value problem to solve is as follows.

$$\psi_{tt} = c^2 \psi_{xx}, \ -\infty < x < \infty, \ -\infty < t < \infty$$

$$\psi(x,0) = \psi_0(x) = \begin{cases} 0 & |x| > a/2 \\ 1/a & |x| < a/2 \end{cases}$$

$$\psi_t(x,0) = 0$$

Since the wave equation is over the whole line ($-\infty < x < \infty$), it can be solved by operator factorization. Bring $c^2 \psi_{xx}$ to the left side.

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} = 0$$

Factor the operator.

$$\left( \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) \psi = 0$$

$$\left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \psi = 0$$

$$\left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \left( \frac{\partial \psi}{\partial t} - c \frac{\partial \psi}{\partial x} \right) = 0$$

Let $u$ be the quantity in the second set of parentheses.

$$\left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u = 0$$

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

As a result of factoring the operator, the wave equation has reduced to a system of first-order PDEs.

$$\begin{cases} \frac{\partial \psi}{\partial t} - c \frac{\partial \psi}{\partial x} = u \\ \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \end{cases}$$

The differential of a function of two variables $h = h(x,t)$ is defined as

$$dh = \frac{\partial h}{\partial t} dt + \frac{\partial h}{\partial x} dx.$$
Divide both sides by \( dt \) to obtain the fundamental relationship between the total derivative of \( h \) and the partial derivatives of \( h \).

\[
\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{dx}{dt} \frac{\partial h}{\partial x}
\]

In light of this, the PDE for \( u \) reduces to the ODE,

\[
\frac{du}{dt} = 0, \tag{1}
\]

along the characteristic curves in the \( xt \)-plane that satisfy

\[
\frac{dx}{dt} = c, \quad x(\xi, 0) = \xi,
\]

where \( \xi \) is a characteristic coordinate. Integrate both sides of equation (2) with respect to \( t \) to solve for \( x(\xi, t) \).

\[x = ct + \xi\]

Now integrate both sides of equation (1) with respect to \( t \).

\[u(x, \xi) = f(\xi)\]

\( f \) is an arbitrary function of the characteristic coordinate \( \xi \). Eliminate \( \xi \) in favor of \( x \) and \( t \).

\[u(x, t) = f(x - ct)\]

Consequently, the PDE for \( \psi \) becomes

\[
\frac{\partial \psi}{\partial t} - c \frac{\partial \psi}{\partial x} = f(x - ct).
\]

It reduces to

\[
\frac{d\psi}{dt} = f(x - ct) \tag{3}
\]

along the characteristic curves in the \( xt \)-plane that satisfy

\[
\frac{dx}{dt} = -c, \quad x(\eta, 0) = \eta,
\]

where \( \eta \) is another characteristic coordinate. Integrate both sides of equation (4) with respect to \( t \) to solve for \( x(\eta, t) \).

\[x = -ct + \eta\]

Now integrate both sides of equation (3) with respect to \( t \).

\[\psi(x, \eta) = \int^t f(x - cs) \, ds + G(\eta)\]

\( G \) is an arbitrary function of the characteristic coordinate \( \eta \). Make the substitution \( r = x - cs \) in the integral.

\[
\psi(x, \eta) = \int^{x-ct} f(r) \left(-\frac{dr}{c}\right) + G(\eta) \\
= F(x - ct) + G(\eta)
\]
$F$ is the integral of $-f/c$, another arbitrary function. Therefore, since $\eta = x + ct$,

$$\psi(x, t) = F(x - ct) + G(x + ct).$$

This is the general solution of the wave equation. Now apply the initial conditions to determine $F$ and $G$.

$$\psi(x, 0) = F(x) + G(x) = \psi_0(x)$$
$$\psi_t(x, 0) = -cF'(x) + cG'(x) = 0$$

Differentiate both sides of the first equation with respect to $x$ and multiply both sides of it by $c$.

$$cF'(x) + cG'(x) = c\psi'_0(x)$$
$$-cF'(x) + cG'(x) = 0$$

Add both sides of each equation to eliminate $F'$.

$$2cG'(x) = c\psi'_0(x)$$

Divide both sides by $2c$.

$$G'(x) = \frac{1}{2}\psi'_0(x)$$

Integrate both sides with respect to $x$, setting the constant of integration to zero.

$$G(x) = \frac{1}{2}\psi_0(x)$$

So then

$$F(x) + G(x) = \psi_0(x) \rightarrow F(x) + \frac{1}{2}\psi_0(x) = \psi_0(x) \rightarrow F(x) = \frac{1}{2}\psi_0(x).$$

What we have actually solved for are $F(w)$ and $G(w)$, where $w$ is any expression we choose.

$$F(x - ct) = \frac{1}{2}\psi_0(x - ct)$$
$$G(x + ct) = \frac{1}{2}\psi_0(x + ct)$$

As a result,

$$\psi(x, t) = F(x - ct) + G(x + ct)$$
$$= \frac{1}{2}\psi_0(x - ct) + \frac{1}{2}\psi_0(x + ct)$$
$$= \frac{1}{2}[\psi_0(x - ct) + \psi_0(x + ct)].$$
Note that

\[ \psi_0(x - ct) = \begin{cases} \frac{1}{a} & |x - ct| < \frac{a}{2} \\ 0 & |x - ct| > \frac{a}{2} \end{cases} = \begin{cases} \frac{1}{a} & -\frac{a}{2} < x - ct < \frac{a}{2} \\ 0 & x - ct < -\frac{a}{2} \\ 0 & x - ct > \frac{a}{2} \end{cases} \]

and

\[ \psi_0(x + ct) = \begin{cases} \frac{1}{a} & |x + ct| < \frac{a}{2} \\ 0 & |x + ct| > \frac{a}{2} \end{cases} = \begin{cases} \frac{1}{a} & -\frac{a}{2} < x + ct < \frac{a}{2} \\ 0 & x + ct < -\frac{a}{2} \\ 0 & x + ct > \frac{a}{2} \end{cases} . \]

Depending what region in the \( xt \)-plane the point \((x, t)\) is chosen, \( \psi(x, t) \) will be different. These regions are obtained by drawing characteristic lines with slopes \( \pm c \) through \((-\frac{a}{2}, 0)\) and \((\frac{a}{2}, 0)\), the boundaries of where the initial condition is nonzero.

---

www.stemjock.com
Region I

Suppose the point \((x, t)\) is chosen in region I.

\[
\begin{align*}
\text{In this case } x - ct &< \frac{a}{2} \text{ and } x + ct > \frac{a}{2}, \text{ so} \\
\psi(x, t) &= \frac{1}{2} \left[ \psi_0(x - ct) + \psi_0(x + ct) \right] \\
&= \frac{1}{2} (0 + 0) \\
&= 0.
\end{align*}
\]

This formula is valid for \( |x| < ct - \frac{a}{2} \).
Region II

Suppose the point \((x, t)\) is chosen in region II.

In this case \(x - ct < -\frac{a}{2}\) and \(-\frac{a}{2} < x + ct < \frac{a}{2}\), so

\[
\psi(x, t) = \frac{1}{2} [\psi_0(x - ct) + \psi_0(x + ct)]
\]

\[
= \frac{1}{2} \left( 0 + \frac{1}{a} \right)
\]

\[
= \frac{1}{2a}.
\]

This formula is valid for \(|\frac{a}{2} - ct| < -x < \frac{a}{2} + ct\).
Region III

Suppose the point \((x, t)\) is chosen in region III.

In this case \(-\frac{a}{2} < x - ct < \frac{a}{2}\) and \(x + ct > \frac{a}{2}\), so

\[
\psi(x, t) = \frac{1}{2} \left[ \psi_0(x - ct) + \psi_0(x + ct) \right] \\
= \frac{1}{2} \left( \frac{1}{a} + 0 \right) \\
= \frac{1}{2a}.
\]

This formula is valid for \(\frac{a}{2} - ct < x < \frac{a}{2} + ct\).
Region IV

Suppose the point \((x, t)\) is chosen in region IV.

In this case \(x - ct < -\frac{a}{2}\) and \(x + ct < -\frac{a}{2}\), so

\[
\psi(x,t) = \frac{1}{2} [\psi_0(x - ct) + \psi_0(x + ct)]
\]

\[
= \frac{1}{2} (0 + 0)
\]

\[= 0.
\]

This formula is valid for \(-x > \frac{a}{2} + c|t|\).
Region V

Suppose the point \((x, t)\) is chosen in region V.

In this case \(-\frac{a}{2} < x - ct < \frac{a}{2}\) and \(-\frac{a}{2} < x + ct < \frac{a}{2}\), so

\[
\psi(x, t) = \frac{1}{2} [\psi_0(x - ct) + \psi_0(x + ct)]
\]

\[
= \frac{1}{2} \left( \frac{1}{a} + \frac{1}{a} \right)
\]

\[
= \frac{1}{a}.
\]

This formula is valid for \(|x| < \frac{a}{2} - c|t|\).
Region VI

Suppose the point \((x, t)\) is chosen in region VI.

In this case \(x - ct > \frac{a}{2}\) and \(x + ct > \frac{a}{2}\), so

\[
\psi(x, t) = \frac{1}{2} [\psi_0(x - ct) + \psi_0(x + ct)]
\]

\[
= \frac{1}{2} (0 + 0)
\]

\[
= 0.
\]

This formula is valid for \(x > \frac{a}{2} + c|t|\).
Region VII

Suppose the point \((x, t)\) is chosen in region VII.

In this case \(-\frac{a}{2} < x - ct < \frac{a}{2}\) and \(x + ct < -\frac{a}{2}\), so

\[
\psi(x, t) = \frac{1}{2} \left[ \psi_0(x - ct) + \psi_0(x + ct) \right]
= \frac{1}{2} \left( \frac{1}{a} + 0 \right)
= \frac{1}{2a}.
\]

This formula is valid for \(\frac{a}{2} + ct < -x < \frac{a}{2} - ct\).
Region VIII

Suppose the point \((x, t)\) is chosen in region VIII.

In this case \(x - ct > \frac{a}{2}\) and \(-\frac{a}{2} < x + ct < \frac{a}{2}\), so

\[
\psi(x, t) = \frac{1}{2} \left[ \psi_0(x - ct) + \psi_0(x + ct) \right]
\]

\[
= \frac{1}{2} \left( 0 + \frac{1}{a} \right)
\]

\[
= \frac{1}{2a}.
\]

This formula is valid for \(\left| \frac{a}{2} + ct \right| < x < \frac{a}{2} - ct\).
Region IX

Suppose the point \((x, t)\) is chosen in region IX.

In this case \(x - ct > \frac{a}{2}\) and \(x + ct < -\frac{a}{2}\), so

\[
\psi(x, t) = \frac{1}{2} [\psi_0(x - ct) + \psi_0(x + ct)]
\]
\[
= \frac{1}{2} (0 + 0)
\]
\[
= 0.
\]

This formula is valid for \(|x| < -ct - \frac{a}{2}\).

Some of the formulas we found can be combined by forming unions of the regions and using absolute value signs. The union of regions I and IX, for example, results in the following formula for \(\psi(x, t)\).

\[
\psi(x, t) = 0 \quad \text{if } |x| < c|t| - \frac{a}{2}
\]

The union of regions IV and VI results in the following formula for \(\psi(x, t)\).

\[
\psi(x, t) = 0 \quad \text{if } |x| > \frac{a}{2} + c|t|
\]

The union of regions II, III, VII, and VIII results in the following formula for \(\psi(x, t)\).

\[
\psi(x, t) = \frac{1}{2a} \quad \text{if } \left| \frac{a}{2} - c|t| \right| < |x| < \frac{a}{2} + c|t|
\]
Therefore,

\[ \psi(x, t) = \begin{cases} 
\frac{1}{a} & \text{if } |x| < \frac{a}{2} - c|t| \\
0 & \text{if } |x| < c|t| - \frac{a}{2} \\
0 & \text{if } |x| > \frac{a}{2} + c|t| \\
\frac{1}{2a} & \text{if } \left| \frac{a}{2} - c|t| \right| < |x| < \frac{a}{2} + c|t| 
\end{cases} . \]

The solution in each part of the \( xt \)-plane is labelled by color.