Exercise 9.6.4

Solve the wave equation, Eq. (9.89), subject to the indicated conditions.

Determine $\psi(x,t)$ given that at $t = 0$ $\psi_0(x) = 0$ for all $x$, but $\partial \psi / \partial t = \sin(x)$.

Solution

The initial value problem to solve is as follows.

$$\psi_{tt} = c^2 \psi_{xx}, \quad -\infty < x < \infty, \quad -\infty < t < \infty$$
$$\psi(x,0) = 0$$
$$\psi_t(x,0) = \sin x$$

Since the wave equation is over the whole line ($-\infty < x < \infty$), it can be solved by operator factorization. Bring $c^2 \psi_{xx}$ to the left side.

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} = 0$$

Factor the operator.

$$\left( \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) \psi = 0$$
$$\left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \psi = 0$$
$$\left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \left( \frac{\partial \psi}{\partial t} - c \frac{\partial \psi}{\partial x} \right) = 0$$

Let $u$ be the quantity in the second set of parentheses.

$$\left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u = 0$$
$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

As a result of factoring the operator, the wave equation has reduced to a system of first-order PDEs.

$$\begin{cases} \frac{\partial \psi}{\partial t} - c \frac{\partial \psi}{\partial x} = u \\ \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \end{cases}$$

The differential of a function of two variables $h = h(x,t)$ is defined as

$$dh = \frac{\partial h}{\partial t} dt + \frac{\partial h}{\partial x} dx.$$
In light of this, the PDE for $u$ reduces to the ODE,

$$\frac{du}{dt} = 0,$$

along the characteristic curves in the $xt$-plane that satisfy

$$\frac{dx}{dt} = c, \quad x(\xi, 0) = \xi,$$

where $\xi$ is a characteristic coordinate. Integrate both sides of equation (2) with respect to $t$ to solve for $x(\xi, t)$.

$$x = ct + \xi$$

Now integrate both sides of equation (1) with respect to $t$.

$$u(x, \xi) = f(\xi)$$

$f$ is an arbitrary function of the characteristic coordinate $\xi$. Eliminate $\xi$ in favor of $x$ and $t$.

$$u(x, t) = f(x - ct)$$

Consequently, the PDE for $\psi$ becomes

$$\frac{\partial \psi}{\partial t} - c \frac{\partial \psi}{\partial x} = f(x - ct).$$

It reduces to

$$\frac{d\psi}{dt} = f(x - ct)$$

along the characteristic curves in the $xt$-plane that satisfy

$$\frac{dx}{dt} = -c, \quad x(\eta, 0) = \eta,$$

where $\eta$ is another characteristic coordinate. Integrate both sides of equation (4) with respect to $t$ to solve for $x(\eta, t)$.

$$x = -ct + \eta$$

Now integrate both sides of equation (3) with respect to $t$.

$$\psi(x, \eta) = \int_{t}^{t} f(x - cs) \, ds + G(\eta)$$

$G$ is an arbitrary function of the characteristic coordinate $\eta$. Make the substitution $r = x - cs$ in the integral.

$$\psi(x, \eta) = \int_{x-ct}^{x} f(r) \left( - \frac{dr}{c} \right) + G(\eta)$$

$$= F(x - ct) + G(\eta)$$

$F$ is the integral of $-f/c$, another arbitrary function. Therefore, since $\eta = x + ct$,

$$\psi(x, t) = F(x - ct) + G(x + ct).$$

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This is the general solution of the wave equation. Now apply the initial conditions to determine $F$ and $G$.

\[
\psi(x,0) = F(x) + G(x) = 0 \\
\psi_t(x,0) = -cF'(x) + cG'(x) = \sin x
\]

Differentiate both sides of the first equation with respect to $x$ and multiply both sides of it by $c$.

\[
cF'(x) + cG'(x) = 0 \\
-cF'(x) + cG'(x) = \sin x
\]

Add both sides of each equation to eliminate $F'$.

\[
2cG'(x) = \sin x
\]

Divide both sides by $2c$.

\[
G'(x) = \frac{1}{2c} \sin x
\]

Integrate both sides with respect to $x$, setting the constant of integration to zero.

\[
G(x) = -\frac{1}{2c} \cos x
\]

So then

\[
F(x) + G(x) = 0 \quad \rightarrow \quad F(x) - \frac{1}{2c} \cos x = 0 \quad \rightarrow \quad F(x) = \frac{1}{2c} \cos x.
\]

What we have actually solved for are $F(w)$ and $G(w)$, where $w$ is any expression we choose.

\[
F(x - ct) = \frac{1}{2c} \cos(x - ct) \\
G(x + ct) = -\frac{1}{2c} \cos(x + ct)
\]

As a result,

\[
\psi(x,t) = F(x - ct) + G(x + ct) \\
= \frac{1}{2c} \cos(x - ct) - \frac{1}{2c} \cos(x + ct) \\
= \frac{1}{2c} \left[ \cos(x - ct) - \cos(x + ct) \right] \\
= \frac{1}{2c} \left[ (\cos x \cos ct + \sin x \sin ct) - (\cos x \cos ct - \sin x \sin ct) \right] \\
= \frac{1}{2c} (2 \sin x \sin ct).
\]

Therefore,

\[
\psi(x,t) = \frac{1}{c} \sin x \sin ct.
\]