Exercise 9.7.1

For a homogeneous spherical solid with constant thermal diffusivity, $K$, and no heat sources, the equation of heat conduction becomes

$$\frac{\partial T(r, t)}{\partial t} = K \nabla^2 T(r, t).$$

Assume a solution of the form

$$T = R(r)\Theta(t)$$

and separate variables. Show that the radial equation may take on the standard form

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} + \alpha^2 r^2 R = 0,$$

and that $\sin \alpha r/r$ and $\cos \alpha r/r$ are its solutions.

[TYPO: $T$ represents the temperature. Use a different variable $\Theta$ for the separated function of $t$.]

Solution

Because the solid is spherical, expand the Laplacian operator in spherical polar coordinates $(r, \theta, \varphi)$, where $\theta$ is the angle from the polar axis.

$$\frac{\partial T(r, t)}{\partial t} = K \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T(r, t)}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T(r, t)}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T(r, t)}{\partial \varphi^2} \right] = 0$$

$T$ is only a function of $r$ and $t$, so the angular derivatives vanish.

$$\frac{\partial T}{\partial t} = K \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right)$$

The equation of heat conduction is linear and homogeneous, so the method of separation of variables can be applied to solve it. Assume a product solution of the form $T(r, t) = R(r)\Theta(t)$ and substitute it into the PDE.

$$\frac{\partial}{\partial t}[R(r)\Theta(t)] = K \frac{\partial}{\partial r} \left[ r^2 \frac{\partial}{\partial r}[R(r)\Theta(t)] \right]$$

$$R \frac{d\Theta}{dt} = \Theta \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right)$$

Divide both sides by $KR(r)\Theta(t)$. (The final answer for $T(r, t)$ will be the same regardless which side $K$ is on.)

$$\frac{1}{K\Theta} \frac{d\Theta}{dt} = \frac{1}{r^2 R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right)$$

The only way a function of $t$ can be equal to a function of $r$ is if both are equal to a constant $\lambda$.

$$\frac{1}{K\Theta} \frac{d\Theta}{dt} = \frac{1}{r^2 R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \lambda$$

www.stemjock.com
As a result of applying the method of separation of variables, the equation of conduction has reduced to two ODEs—one in $r$ and one in $t$.

\[
\begin{align*}
\frac{1}{K \Theta} \frac{d \Theta}{dt} &= \lambda \\
\frac{1}{r^2 R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) &= \lambda
\end{align*}
\]

Solve the first ODE for $\Theta$.

\[
\frac{d \Theta}{dt} = K \lambda \Theta
\]

The general solution is written in terms of the exponential function.

\[
\Theta(t) = C_1 e^{K\lambda t}
\]

In order for $T(r, t)$ to remain bounded as $t \to \infty$, we require that $\lambda$ be either zero or negative. Suppose first that $\lambda$ is zero: $\lambda = 0$. The ODE for $R$ becomes

\[
\frac{1}{r^2 R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = 0.
\]

Multiply both sides by $r^2 R$.

\[
\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = 0.
\]

Integrate both sides with respect to $r$.

\[
r^2 \frac{dR}{dr} = C_2
\]

Divide both sides by $r^2$.

\[
\frac{dR}{dr} = \frac{C_2}{r^2}
\]

Integrate both sides with respect to $r$ once more.

\[
R(r) = -\frac{C_2}{r} + C_3
\]

Note that this is the steady-state temperature profile in a spherical geometry. With two boundary conditions, one could determine the constants, $C_2$ and $C_3$. Suppose secondly that $\lambda$ is negative: $\lambda = -\alpha^2$. The ODE for $R$ becomes

\[
\frac{1}{r^2 R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = -\alpha^2.
\]

Multiply both sides by $r^2 R$.

\[
\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = -\alpha^2 r^2 R
\]

Use the product rule to expand the left side.

\[
r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} = -\alpha^2 r^2 R
\]

The radial equation is thus

\[
r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} + \alpha^2 r^2 R = 0.
\]

www.stemjock.com
Make the change of variables,

\[ R = \frac{W}{r}. \]

Find the derivatives of \( R \) in terms of this new variable.

\[
\frac{dR}{dr} = -\frac{W}{r^2} + \frac{W'}{r} \\
\frac{d^2R}{dr^2} = \frac{2}{r^3}W - \frac{W'}{r^2} - \frac{W''}{r} + \frac{W'''}{r} = \frac{2}{r^3}W - \frac{2}{r^2}W' + \frac{1}{r}W''
\]

Substitute these formulas into the radial equation to obtain an ODE for \( W \).

\[
r^2 \left( \frac{2}{r^3} W - \frac{2}{r^2} W' + \frac{1}{r} W'' \right) + 2r \left( -\frac{W}{r^2} + \frac{W'}{r} \right) + \alpha^2 r^2 \left( \frac{W}{r} \right) = 0
\]

\[
\frac{2}{r} W - 2W' + rW'' - \frac{2}{r} W + 2W' + \alpha^2 rW = 0
\]

\[
rW'' + \alpha^2 rW = 0
\]

Divide both sides by \( r \).

\[
W'' + \alpha^2 W = 0
\]

The general solution is written in terms of sine and cosine.

\[
W(r) = C_4 \cos \alpha r + C_5 \sin \alpha r
\]

Therefore, since \( R = W/r \),

\[
R(r) = C_4 \frac{\cos \alpha r}{r} + C_5 \frac{\sin \alpha r}{r}.
\]