Exercise 9.7.2

Separate variables in the thermal diffusion equation of Exercise 9.7.1 in circular cylindrical coordinates. Assume that you can neglect end effects and take $T = T(\rho, t)$.

Solution

The thermal diffusion equation of Exercise 9.7.1 is

$$\frac{\partial T}{\partial t} = K \nabla^2 T.$$  

Expand the Laplacian operator in circular cylindrical coordinates $(\rho, \varphi, z)$.

$$\frac{\partial T(\rho, t)}{\partial t} = K \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial T(\rho, t)}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 T(\rho, t)}{\partial \varphi^2} + \frac{\partial^2 T(\rho, t)}{\partial z^2} \right] = 0$$

$T$ is only a function of $\rho$ and $t$, so the angular derivatives vanish.

$$\frac{\partial T}{\partial t} = K \frac{\partial}{\partial \rho} \left( \rho \frac{\partial T}{\partial \rho} \right)$$

The equation of heat conduction is linear and homogeneous, so the method of separation of variables can be applied to solve it. Assume a product solution of the form $T(\rho, t) = P(\rho) \Theta(t)$ and substitute it into the PDE.

$$\frac{\partial}{\partial t} [P(\rho) \Theta(t)] = K \frac{\partial}{\partial \rho} \left[ \rho \frac{\partial [P(\rho) \Theta(t)]}{\partial \rho} \right]$$

$$P \frac{d \Theta}{dt} = \Theta K \frac{d}{d \rho} \left( \rho \frac{d P}{d \rho} \right)$$

Divide both sides by $K P(\rho) \Theta(t)$. (The final answer for $T(\rho, t)$ will be the same regardless which side $K$ is on.)

$$\frac{1}{K \Theta} \frac{d \Theta}{dt} = \frac{1}{\rho P} \frac{d}{d \rho} \left( \rho \frac{d P}{d \rho} \right)$$

The only way a function of $t$ can be equal to a function of $\rho$ is if both are equal to a constant $\lambda$.

$$\frac{1}{K \Theta} \frac{d \Theta}{dt} = \frac{1}{\rho P} \frac{d}{d \rho} \left( \rho \frac{d P}{d \rho} \right) = \lambda$$

As a result of applying the method of separation of variables, the equation of conduction has reduced to two ODEs—one in $\rho$ and one in $t$.

$$\frac{1}{K \Theta} \frac{d \Theta}{dt} = \lambda$$

$$\frac{1}{\rho P} \frac{d}{d \rho} \left( \rho \frac{d P}{d \rho} \right) = \lambda$$

Solve the first ODE for $\Theta$.

$$\frac{d \Theta}{dt} = K \lambda \Theta$$
The general solution is written in terms of the exponential function.

\[ \Theta(t) = C_1 e^{K \lambda t} \]

In order for \( T(\rho, t) \) to remain bounded as \( t \to \infty \), we require that \( \lambda \) be either zero or negative. Suppose first that \( \lambda \) is zero: \( \lambda = 0 \). The ODE for \( P \) becomes

\[ \frac{1}{\rho P} \frac{d}{d\rho} \left( \rho \frac{dP}{d\rho} \right) = 0. \]

Multiply both sides by \( \rho P \).

\[ \frac{d}{d\rho} \left( \rho \frac{dP}{d\rho} \right) = 0. \]

Integrate both sides with respect to \( \rho \).

\[ \rho \frac{dP}{d\rho} = C_2 \]

Divide both sides by \( \rho \).

\[ \frac{dP}{d\rho} = \frac{C_2}{\rho} \]

Integrate both sides with respect to \( \rho \) once more.

\[ P(\rho) = C_2 \ln \rho + C_3 \]

Note that this is the steady-state temperature profile in a cylindrical geometry. With two boundary conditions, one could determine the constants, \( C_2 \) and \( C_3 \). Suppose secondly that \( \lambda \) is negative: \( \lambda = -\alpha^2 \). The ODE for \( P \) becomes

\[ \frac{1}{\rho P} \frac{d}{d\rho} \left( \rho \frac{dP}{d\rho} \right) = -\alpha^2. \]

Multiply both sides by \( \rho^2 P \).

\[ \rho \frac{d}{d\rho} \left( \rho \frac{dP}{d\rho} \right) = -\alpha^2 \rho^2 P \]

Use the product rule to expand the left side.

\[ \rho \left( \rho \frac{d^2P}{d\rho^2} + \frac{dP}{d\rho} \right) = -\alpha^2 \rho^2 P \]

The radial equation is thus

\[ \rho^2 \frac{d^2P}{d\rho^2} + \rho \frac{dP}{d\rho} + \alpha^2 \rho^2 P = 0, \]

which is known as the Bessel equation of order zero. Its general solution is written in terms of \( J_0 \) and \( Y_0 \), the Bessel functions of the first and second kind, respectively.

\[ P(\rho) = C_4 J_0(\alpha \rho) + C_5 Y_0(\alpha \rho) \]