Exercise 9.7.3

Solve the PDE

$$\frac{\partial \psi}{\partial t} = a^2 \frac{\partial^2 \psi}{\partial x^2},$$

to obtain $\psi(x,t)$ for a rod of infinite extent (in both the $+x$ and $-x$ directions), with a heat pulse at time $t = 0$ that corresponds to $\psi_0(x) = A\delta(x)$.

Solution

The initial value problem to solve is

$$\begin{aligned} &\frac{\partial \psi}{\partial t} = a^2 \frac{\partial^2 \psi}{\partial x^2}, \quad -\infty < x < \infty,\ t > 0 \\ &\psi(x,0) = A\delta(x). \end{aligned}$$

Consider the similar problem,

$$\begin{aligned} &\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty,\ t > 0 \\ &u(x,0) = H(x), \end{aligned}$$

where $H(x)$ is the Heaviside function, defined to be 0 for $x < 0$ and 1 for $x > 0$. The similarity method (also known as combination of variables) will be used here: assuming that u is dimensionless, the variables in the solution must be arranged as

$$u(x,t) = f\left(\frac{x}{\sqrt{a^2 t}}\right)$$

for it to be dimensionally consistent. Note that x is distance, a^2 is distance^2/time, and t is time. Also note that the variables could be combined as $x^2/(a^2 t)$, but it leads to a more complicated ODE for f. Substitute this function for u into the PDE.

$$\begin{aligned} \frac{\partial}{\partial t} f\left(\frac{x}{\sqrt{a^2 t}}\right) &= a^2 \frac{\partial^2}{\partial x^2} f\left(\frac{x}{\sqrt{a^2 t}}\right) \\ \left(-\frac{x}{2\sqrt{a^2 t^3}}\right) f'\left(\frac{x}{\sqrt{a^2 t}}\right) &= a^2 \left(\frac{1}{\sqrt{a^2 t}}\right)^2 f''\left(\frac{x}{\sqrt{a^2 t}}\right) \\ -\frac{x}{2\sqrt{a^2 t^3}} f'\left(\frac{x}{\sqrt{a^2 t}}\right) &= \frac{1}{t} f''\left(\frac{x}{\sqrt{a^2 t}}\right) \end{aligned}$$

Multiply both sides by t.

$$-\frac{x}{2\sqrt{a^2 t}} f'\left(\frac{x}{\sqrt{a^2 t}}\right) = f''\left(\frac{x}{\sqrt{a^2 t}}\right)$$

Letting $\xi = x/\sqrt{a^2 t}$, the ODE that f satisfies is

$$f''(\xi) = -\frac{\xi}{2} f'(\xi).$$

Divide both sides by $f'(\xi)$.

$$\frac{f''}{f'} = -\frac{\xi}{2}$$

Rewrite the left side as $d/d\xi(\ln|f'|)$ using the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{d\xi}(\ln|f'|) = -\frac{\xi}{2}$$

Integrate both sides with respect to ξ.

$$\ln|f'| = -\frac{\xi^2}{4} + C_1$$

Exponentiate both sides.

$$\begin{aligned} |f'| &= e^{-\xi^2/4+C_1} \\ &= e^{C_1}e^{-\xi^2/4} \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$f' = \pm e^{C_1}e^{-\xi^2/4}$$

Use a new constant C_2 for $\pm e^{C_1}$.

$$f' = C_2 e^{-\xi^2/4}$$

Integrate both sides with respect to ξ once more.

$$f(\xi) = C_2 \int_0^{\xi} e^{-s^2/4}\, ds + C_3$$

The lower limit of the integral has been arbitrarily set to zero; C_3 will be adjusted to account for whatever choice we make. As a result,

$$u(x,t) = C_2 \int_0^{\frac{x}{\sqrt{a^2t}}} e^{-s^2/4}\, ds + C_3.$$

The constants are determined by using the initial condition $u(x,0) = H(x)$. If $t = 0$, then the upper limit becomes ∞ or $-\infty$, depending whether x is positive or negative, respectively.

$$u(x,0) = \begin{cases} C_2 \displaystyle\int_0^{-\infty} e^{-s^2/4}\, ds + C_3 = 0 & \text{if } x < 0 \\ C_2 \displaystyle\int_0^{\infty} e^{-s^2/4}\, ds + C_3 = 1 & \text{if } x > 0 \end{cases}$$

Evaluate the integrals and solve the system of equations for C_2 and C_3.

$$\begin{cases} -\sqrt{\pi}C_2 + C_3 = 0 \\ \sqrt{\pi}C_2 + C_3 = 1 \end{cases} \quad \rightarrow \quad \begin{cases} C_2 = \dfrac{1}{2\sqrt{\pi}} \\ C_3 = \dfrac{1}{2} \end{cases}$$

Consequently,

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^{\frac{x}{a\sqrt{t}}} e^{-s^2/4}\, ds + \frac{1}{2}.$$

Any constant multiple of a solution to the heat equation is also a solution to the heat equation. In addition, any derivative of a solution to the heat equation is also a solution to the heat equation. So then

$$\psi(x,t) = A\frac{\partial}{\partial x}u(x,t),$$

since

$$A\frac{d}{dx}H(x) = A\delta(x).$$

Substitute the formula for u and simplify.

$$\begin{aligned}
\psi(x,t) &= A\frac{\partial}{\partial x}\left(\frac{1}{2\sqrt{\pi}}\int_0^{\frac{x}{a\sqrt{t}}} e^{-s^2/4}\,ds + \frac{1}{2}\right) \\
&= \frac{A}{2\sqrt{\pi}}\frac{\partial}{\partial x}\int_0^{\frac{x}{a\sqrt{t}}} e^{-s^2/4}\,ds \\
&= \frac{A}{2\sqrt{\pi}}\left(\frac{1}{a\sqrt{t}}\right)\exp\left[-\frac{1}{4}\left(\frac{x}{a\sqrt{t}}\right)^2\right]
\end{aligned}$$

Therefore,

$$\psi(x,t) = \frac{A}{\sqrt{4\pi a^2 t}}\exp\left(-\frac{x^2}{4a^2 t}\right).$$