Exercise 3

A mathematical plane surface of area S has an orientation given by a unit normal vector $\mathbf{n}$, pointing downstream of the surface. A fluid of density ρ flows through this surface with a velocity $\mathbf{v}$. Show that the mass rate of flow through the surface is $w = \rho(\mathbf{n} \cdot \mathbf{v})S$.

Solution

The mass rate of flow with respect to time is dm/dt.

$$w = \frac{dm}{dt}$$

Mass is the product of fluid density ρ and volume V.

$$w = \frac{d}{dt}(\rho V)$$

If we assume that density is constant, then it can be pulled in front of the derivative.

$$w = \rho \frac{dV}{dt}$$

dV/dt represents the volumetric flow rate, that is, the amount of fluid that flows through the surface per unit time. It is equal to the product of the surface's area S and the component of velocity in the direction of the surface's normal vector $\mathbf{v} \cdot \mathbf{n}$.

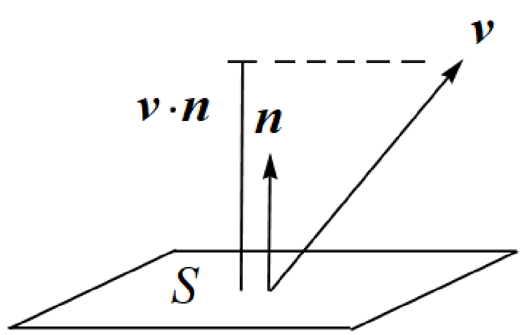

Figure 1: $\mathbf{v} \cdot \mathbf{n}$ represents the component of the velocity in the normal direction.

$$w = \rho S \mathbf{v} \cdot \mathbf{n}$$

The dot product is commutative. Therefore,

$$w = \rho(\mathbf{n} \cdot \mathbf{v})S.$$