Exercise 7

Verify that $([\mathbf{v} \times \mathbf{w}] \cdot [\mathbf{v} \times \mathbf{w}]) + (\mathbf{v} \cdot \mathbf{w})^2 = v^2 w^2$ (the "identity of Lagrange").

Solution

The left side of Lagrange's identity can be written as

$$
\begin{aligned}
([\mathbf{v} \times \mathbf{w}] \cdot [\mathbf{v} \times \mathbf{w}]) + (\mathbf{v} \cdot \mathbf{w})^2 &= |\mathbf{v} \times \mathbf{w}|^2 + (\mathbf{v} \cdot \mathbf{w})^2. \\
&= (|\mathbf{v}||\mathbf{w}| \sin \phi_{\boldsymbol{vw}})^2 + (|\mathbf{v}||\mathbf{w}| \cos \phi_{\boldsymbol{vw}})^2 \\
&= |\mathbf{v}|^2 |\mathbf{w}|^2 \sin^2 \phi_{\boldsymbol{vw}} + |\mathbf{v}|^2 |\mathbf{w}|^2 \cos^2 \phi_{\boldsymbol{vw}} \\
&= |\mathbf{v}|^2 |\mathbf{w}|^2 (\sin^2 \phi_{\boldsymbol{vw}} + \cos^2 \phi_{\boldsymbol{vw}}) \\
&= |\mathbf{v}|^2 |\mathbf{w}|^2 \\
&= v^2 w^2
\end{aligned}
$$

Therefore, the identity of Lagrange is verified.