Exercise 6

The kinetic energy of rotation of the rigid structure in Exercise 5 is

$$K = \sum_v \frac{1}{2} m_v \left(\dot{\mathbf{R}}_v \cdot \dot{\mathbf{R}}_v \right)$$

where $\dot{\mathbf{R}}_v = [\mathbf{W} \times \mathbf{R}_v]$ is the velocity of the vth particle. Show that

$$K = \frac{1}{2} (\mathbf{\Phi} : \mathbf{WW})$$

Solution

Start off by substituting $\mathbf{W} \times \mathbf{R}_v$ for $\dot{\mathbf{R}}_v$ in the expression for K.

$$K = \sum_{v=1}^{N} \frac{1}{2} m_v \left(\dot{\mathbf{R}}_v \cdot [\mathbf{W} \times \mathbf{R}_v] \right)$$

The point of only substituting it into the second $\dot{\mathbf{R}}_v$ is so that we can use the following triple-product vector identity.

$$\mathbf{A} \cdot [\mathbf{B} \times \mathbf{C}] = \mathbf{B} \cdot [\mathbf{C} \times \mathbf{A}]$$

Doing so gives us

$$K = \sum_{v=1}^{N} \frac{1}{2} m_v \left(\mathbf{W} \cdot [\mathbf{R}_v \times \dot{\mathbf{R}}_v] \right).$$

Replace $\dot{\mathbf{R}}_v$ with $\mathbf{W} \times \mathbf{R}_v$.

$$K = \sum_{v=1}^{N} \frac{1}{2} m_v \left(\mathbf{W} \cdot [\mathbf{R}_v \times [\mathbf{W} \times \mathbf{R}_v]] \right)$$

Now make use of the BAC-CAB vector identity.

$$\mathbf{A} \times [\mathbf{B} \times \mathbf{C}] = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

The equation becomes

$$K = \sum_{v=1}^{N} \frac{1}{2} m_v \left(\mathbf{W} \cdot [\mathbf{W}(\mathbf{R}_v \cdot \mathbf{R}_v) - \mathbf{R}_v(\mathbf{R}_v \cdot \mathbf{W})] \right).$$

Move $\mathbf{W}$ to the right side and bring 1/2 out of the sum.

$$K = \frac{1}{2} \sum_{v=1}^{N} m_v \left([(\mathbf{R}_v \cdot \mathbf{R}_v)\mathbf{W} - \mathbf{R}_v(\mathbf{R}_v \cdot \mathbf{W})] \cdot \mathbf{W} \right)$$

We can write $\mathbf{W}$ in terms of the unit tensor $\boldsymbol{\delta}$ as $\boldsymbol{\delta}\cdot\mathbf{W}$. This will be shown now.

$$\begin{aligned}\boldsymbol{\delta}\cdot\mathbf{W} = \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_i\boldsymbol{\delta}_j\delta_{ij}\right)\cdot\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k W_k\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_i(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)\delta_{ij}W_k &= \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)W_k \\ &= \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\delta_{jk}W_k \\ &= \sum_{k=1}^{3}\boldsymbol{\delta}_k W_k \\ &= \mathbf{W}\end{aligned}$$

Hence,

$$K = \frac{1}{2}\sum_{v=1}^{N} m_v\left([(\mathbf{R}_v\cdot\mathbf{R}_v)[\boldsymbol{\delta}\cdot\mathbf{W}] - \mathbf{R}_v(\mathbf{R}_v\cdot\mathbf{W})]\cdot\mathbf{W}\right).$$

Factor out $\mathbf{W}$.

$$K = \frac{1}{2}\sum_{v=1}^{N} m_v\left([\{(\mathbf{R}_v\cdot\mathbf{R}_v)\boldsymbol{\delta} - \mathbf{R}_v\mathbf{R}_v\}\cdot\mathbf{W}]\cdot\mathbf{W}\right)$$

Bring m_v and the sum inside the two dot products.

$$K = \frac{1}{2}\left(\left[\sum_{v=1}^{N} m_v\{(\mathbf{R}_v\cdot\mathbf{R}_v)\boldsymbol{\delta} - \mathbf{R}_v\mathbf{R}_v\}\cdot\mathbf{W}\right]\cdot\mathbf{W}\right)$$

Note that

$$\boldsymbol{\Phi} = \sum_{v=1}^{N} m_v\left\{(\mathbf{R}_v\cdot\mathbf{R}_v)\boldsymbol{\delta} - \mathbf{R}_v\mathbf{R}_v\right\}$$

is the moment-of-inertia tensor, so the expression for the kinetic energy simplifies to

$$K = \frac{1}{2}([\boldsymbol{\Phi}\cdot\mathbf{W}]\cdot\mathbf{W}).$$

This can be written with the double dot product by considering the dyadic product $\mathbf{WW}$. Therefore,

$$K = \frac{1}{2}\left(\boldsymbol{\Phi} : \mathbf{WW}\right).$$