Exercise 6

The kinetic energy of rotation of the rigid structure in Exercise 5 is

\[ K = \sum_v \frac{1}{2} m_v \left( \dot{R}_v \cdot \dot{R}_v \right) \]

where \( \dot{R}_v = [W \times R_v] \) is the velocity of the \( v \)-th particle. Show that

\[ K = \frac{1}{2} (\Phi : WW) \]

Solution

Start off by substituting \( W \times R_v \) for \( \dot{R}_v \) in the expression for \( K \).

\[ K = \sum_{v=1}^N \frac{1}{2} m_v \left( \dot{R}_v \cdot [W \times R_v] \right) \]

The point of only substituting it into the second \( \dot{R}_v \) is so that we can use the following triple-product vector identity.

\[ A \cdot (B \times C) = B \cdot (C \times A) \]

Doing so gives us

\[ K = \sum_{v=1}^N \frac{1}{2} m_v \left( W \cdot [R_v \times \dot{R}_v] \right). \]

Replace \( \dot{R}_v \) with \( W \times R_v \).

\[ K = \sum_{v=1}^N \frac{1}{2} m_v \left( W \cdot [R_v \times (W \times R_v)] \right). \]

Now make use of the BAC-CAB vector identity.

\[ A \times [B \times C] = B(A \cdot C) - C(A \cdot B) \]

The equation becomes

\[ K = \sum_{v=1}^N \frac{1}{2} m_v \left( W \cdot [W(R_v \cdot R_v) - R_v(R_v \cdot W)] \right). \]

Move \( W \) to the right side and bring \( 1/2 \) out of the sum.

\[ K = \frac{1}{2} \sum_{v=1}^N m_v \left( [R_v \cdot R_v]W - R_v(R_v \cdot W) \right) \cdot W \]
We can write $W$ in terms of the unit tensor $\delta$ as $\delta \cdot W$. This will be shown now.

$$\delta \cdot W = \left( \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_i \delta_j \delta_{ij} \right) \cdot \left( \sum_{k=1}^{3} \delta_k W_k \right) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \delta_i (\delta_j \cdot \delta_k) \delta_{ij} W_k = \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_j (\delta_j \cdot \delta_k) W_k$$

$$= \sum_{j=1}^{3} \sum_{k=1}^{3} \delta_j \delta_{jk} W_k$$

$$= \sum_{k=1}^{3} \delta_k W_k$$

$$= W$$

Hence,

$$K = \frac{1}{2} \sum_{v=1}^{N} m_v \left( \left( \left( R_v \cdot R_v \right) \delta \cdot W \right) - R_v \left( R_v \cdot W \right) \right) \cdot W \right)$$

Factor out $W$.

$$K = \frac{1}{2} \sum_{v=1}^{N} m_v \left( \left( \left( R_v \cdot R_v \right) \delta - R_v R_v \right) \cdot W \right) \cdot W \right)$$

Bring $m_v$ and the sum inside the two dot products.

$$K = \frac{1}{2} \left( \sum_{v=1}^{N} m_v \left( \left( R_v \cdot R_v \right) \delta - R_v R_v \right) \cdot W \right) \cdot W \right)$$

Note that

$$\Phi = \sum_{v=1}^{N} m_v \left( \left( R_v \cdot R_v \right) \delta - R_v R_v \right)$$

is the moment-of-inertia tensor, so the expression for the kinetic energy simplifies to

$$K = \frac{1}{2} (\Phi \cdot W) \cdot W \right)$$

This can be written with the double dot product by considering the dyadic product $WW$.

Therefore,

$$K = \frac{1}{2} (\Phi : WW)$$

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