Exercise 10

Write out in full in Cartesian coordinates

$$\textbf{(a)} \quad \frac{\partial}{\partial t}\rho\mathbf{v} = -[\nabla \cdot \rho\mathbf{v}\mathbf{v}] - \nabla p - [\nabla \cdot \boldsymbol{\tau}] + \rho\mathbf{g}$$

$$\textbf{(b)} \quad \boldsymbol{\tau} = -\mu\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger - \frac{2}{3}(\nabla\cdot\mathbf{v})\boldsymbol{\delta}\right]$$

Solution

Part (a)

The equation we're interested in here is the following.

$$\frac{\partial}{\partial t}\rho\mathbf{v} = -[\nabla \cdot \rho\mathbf{v}\mathbf{v}] - \nabla p - [\nabla \cdot \boldsymbol{\tau}] + \rho\mathbf{g} \tag{1}$$

Writing out the left-hand side, we get

$$\frac{\partial}{\partial t}\rho\mathbf{v} = \frac{\partial}{\partial t}\rho\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i v_i\right) = \sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial t}(\rho v_i) = \boldsymbol{\delta}_1\frac{\partial}{\partial t}(\rho v_1) + \boldsymbol{\delta}_2\frac{\partial}{\partial t}(\rho v_2) + \boldsymbol{\delta}_3\frac{\partial}{\partial t}(\rho v_3).$$

Writing out each of the terms on the right-hand side, we get

$$\begin{aligned}
-[\nabla\cdot\rho\mathbf{v}\mathbf{v}] &= -\left[\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\rho\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j v_j\right)\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k v_k\right)\right] \\
&= -\left[\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\rho\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\boldsymbol{\delta}_k v_j v_k\right)\right] \\
&= -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j)\boldsymbol{\delta}_k\frac{\partial}{\partial x_i}(\rho v_j v_k) \\
&= -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\delta_{ij}\boldsymbol{\delta}_k\frac{\partial}{\partial x_i}(\rho v_j v_k) \\
&= -\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\frac{\partial}{\partial x_j}(\rho v_j v_k) \\
&= -\sum_{k=1}^{3}\boldsymbol{\delta}_k\left[\sum_{j=1}^{3}\frac{\partial}{\partial x_j}(\rho v_j v_k)\right] \\
&= -\sum_{k=1}^{3}\boldsymbol{\delta}_k\left[\frac{\partial}{\partial x_1}(\rho v_1 v_k) + \frac{\partial}{\partial x_2}(\rho v_2 v_k) + \frac{\partial}{\partial x_3}(\rho v_3 v_k)\right] \\
&= -\boldsymbol{\delta}_1\left[\frac{\partial}{\partial x_1}(\rho v_1 v_1) + \frac{\partial}{\partial x_2}(\rho v_2 v_1) + \frac{\partial}{\partial x_3}(\rho v_3 v_1)\right] \\
&\quad - \boldsymbol{\delta}_2\left[\frac{\partial}{\partial x_1}(\rho v_1 v_2) + \frac{\partial}{\partial x_2}(\rho v_2 v_2) + \frac{\partial}{\partial x_3}(\rho v_3 v_2)\right] \\
&\quad - \boldsymbol{\delta}_3\left[\frac{\partial}{\partial x_1}(\rho v_1 v_3) + \frac{\partial}{\partial x_2}(\rho v_2 v_3) + \frac{\partial}{\partial x_3}(\rho v_3 v_3)\right]
\end{aligned}$$

$$\begin{aligned}
-\nabla p &= -\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) p \\
&= -\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial p}{\partial x_i} \\
&= -\boldsymbol{\delta}_1 \frac{\partial p}{\partial x_1} - \boldsymbol{\delta}_2 \frac{\partial p}{\partial x_2} - \boldsymbol{\delta}_3 \frac{\partial p}{\partial x_3}
\end{aligned}$$

$$\begin{aligned}
-[\nabla \cdot \boldsymbol{\tau}] &= -\left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \boldsymbol{\delta}_k \tau_{jk}\right)\right] \\
&= -\left[\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \boldsymbol{\delta}_k \frac{\partial \tau_{jk}}{\partial x_i}\right] \\
&= -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \delta_{ij} \boldsymbol{\delta}_k \frac{\partial \tau_{jk}}{\partial x_i} \\
&= -\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial \tau_{jk}}{\partial x_j} \\
&= -\sum_{k=1}^{3} \boldsymbol{\delta}_k \left(\sum_{j=1}^{3} \frac{\partial \tau_{jk}}{\partial x_j}\right) \\
&= -\sum_{k=1}^{3} \boldsymbol{\delta}_k \left(\frac{\partial \tau_{1k}}{\partial x_1} + \frac{\partial \tau_{2k}}{\partial x_2} + \frac{\partial \tau_{3k}}{\partial x_3}\right) \\
&= -\boldsymbol{\delta}_1 \left(\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3}\right) \\
&\quad - \boldsymbol{\delta}_2 \left(\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3}\right) \\
&\quad - \boldsymbol{\delta}_3 \left(\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3}\right)
\end{aligned}$$

$$\begin{aligned}
\rho \mathbf{g} &= \rho \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i g_i\right) \\
&= \sum_{i=1}^{3} \boldsymbol{\delta}_i \rho g_i \\
&= \boldsymbol{\delta}_1 \rho g_1 + \boldsymbol{\delta}_2 \rho g_2 + \boldsymbol{\delta}_3 \rho g_3.
\end{aligned}$$

Substitute the final forms of each expression into equation (1).

$$\begin{aligned}
\boldsymbol{\delta}_1 \frac{\partial}{\partial t}(\rho v_1) \quad &= -\boldsymbol{\delta}_1 \left[\frac{\partial}{\partial x_1}(\rho v_1 v_1) + \frac{\partial}{\partial x_2}(\rho v_2 v_1) + \frac{\partial}{\partial x_3}(\rho v_3 v_1)\right] - \boldsymbol{\delta}_1 \frac{\partial p}{\partial x_1} - \boldsymbol{\delta}_1 \left(\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3}\right) + \boldsymbol{\delta}_1 \rho g_1 \\
+\boldsymbol{\delta}_2 \frac{\partial}{\partial t}(\rho v_2) \quad & \quad - \boldsymbol{\delta}_2 \left[\frac{\partial}{\partial x_1}(\rho v_1 v_2) + \frac{\partial}{\partial x_2}(\rho v_2 v_2) + \frac{\partial}{\partial x_3}(\rho v_3 v_2)\right] - \boldsymbol{\delta}_2 \frac{\partial p}{\partial x_2} - \boldsymbol{\delta}_2 \left(\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3}\right) + \boldsymbol{\delta}_2 \rho g_2 \\
+\boldsymbol{\delta}_3 \frac{\partial}{\partial t}(\rho v_3) \quad & \quad - \boldsymbol{\delta}_3 \left[\frac{\partial}{\partial x_1}(\rho v_1 v_3) + \frac{\partial}{\partial x_2}(\rho v_2 v_3) + \frac{\partial}{\partial x_3}(\rho v_3 v_3)\right] - \boldsymbol{\delta}_3 \frac{\partial p}{\partial x_3} - \boldsymbol{\delta}_3 \left(\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3}\right) + \boldsymbol{\delta}_3 \rho g_3
\end{aligned}$$

The component of $\boldsymbol{\delta}_1$ on the left side must be equal to the component of $\boldsymbol{\delta}_1$ on the right side, and the same is true for the components of $\boldsymbol{\delta}_2$ and $\boldsymbol{\delta}_3$. Therefore, equation (1) is equivalent to the following three equations in Cartesian coordinates.

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho v_1) &= -\left[\frac{\partial}{\partial x_1}(\rho v_1 v_1) + \frac{\partial}{\partial x_2}(\rho v_2 v_1) + \frac{\partial}{\partial x_3}(\rho v_3 v_1)\right] - \frac{\partial p}{\partial x_1} - \left(\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3}\right) + \rho g_1 \\
\frac{\partial}{\partial t}(\rho v_2) &= -\left[\frac{\partial}{\partial x_1}(\rho v_1 v_2) + \frac{\partial}{\partial x_2}(\rho v_2 v_2) + \frac{\partial}{\partial x_3}(\rho v_3 v_2)\right] - \frac{\partial p}{\partial x_2} - \left(\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3}\right) + \rho g_2 \\
\frac{\partial}{\partial t}(\rho v_3) &= -\left[\frac{\partial}{\partial x_1}(\rho v_1 v_3) + \frac{\partial}{\partial x_2}(\rho v_2 v_3) + \frac{\partial}{\partial x_3}(\rho v_3 v_3)\right] - \frac{\partial p}{\partial x_3} - \left(\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3}\right) + \rho g_3
\end{aligned}$$

Part (b)

The equation we're interested in here is the following.

$$\boldsymbol{\tau} = -\mu \nabla \mathbf{v} - \mu (\nabla \mathbf{v})^\dagger + \frac{2}{3}\mu (\nabla \cdot \mathbf{v}) \boldsymbol{\delta} \tag{2}$$

Writing out the left-hand side, we get

$$\begin{aligned}
\boldsymbol{\tau} &= \sum_{i=1}^{3} \sum_{j=1}^{3} \boldsymbol{\delta}_i \boldsymbol{\delta}_j \tau_{ij} \\
&= \sum_{i=1}^{3} \boldsymbol{\delta}_i \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \tau_{ij}\right) \\
&= \sum_{i=1}^{3} \boldsymbol{\delta}_i (\boldsymbol{\delta}_1 \tau_{i1} + \boldsymbol{\delta}_2 \tau_{i2} + \boldsymbol{\delta}_3 \tau_{i3}) \\
&= \boldsymbol{\delta}_1 (\boldsymbol{\delta}_1 \tau_{11} + \boldsymbol{\delta}_2 \tau_{12} + \boldsymbol{\delta}_3 \tau_{13}) \\
&\quad + \boldsymbol{\delta}_2 (\boldsymbol{\delta}_1 \tau_{21} + \boldsymbol{\delta}_2 \tau_{22} + \boldsymbol{\delta}_3 \tau_{23}) \\
&\quad + \boldsymbol{\delta}_3 (\boldsymbol{\delta}_1 \tau_{31} + \boldsymbol{\delta}_2 \tau_{32} + \boldsymbol{\delta}_3 \tau_{33}) \\
&= \boldsymbol{\delta}_1 \boldsymbol{\delta}_1 \tau_{11} + \boldsymbol{\delta}_1 \boldsymbol{\delta}_2 \tau_{12} + \boldsymbol{\delta}_1 \boldsymbol{\delta}_3 \tau_{13} \\
&\quad + \boldsymbol{\delta}_2 \boldsymbol{\delta}_1 \tau_{21} + \boldsymbol{\delta}_2 \boldsymbol{\delta}_2 \tau_{22} + \boldsymbol{\delta}_2 \boldsymbol{\delta}_3 \tau_{23} \\
&\quad + \boldsymbol{\delta}_3 \boldsymbol{\delta}_1 \tau_{31} + \boldsymbol{\delta}_3 \boldsymbol{\delta}_2 \tau_{32} + \boldsymbol{\delta}_3 \boldsymbol{\delta}_3 \tau_{33}.
\end{aligned}$$

Writing out each of the terms on the right-hand side, we get

$$\begin{aligned}
-\mu\nabla\mathbf{v} &= -\mu\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j v_j\right)\\
&= -\mu\sum_{i=1}^{3}\boldsymbol{\delta}_i\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial v_j}{\partial x_i}\right)\\
&= -\mu\sum_{i=1}^{3}\boldsymbol{\delta}_i\left(\boldsymbol{\delta}_1\frac{\partial v_1}{\partial x_i}+\boldsymbol{\delta}_2\frac{\partial v_2}{\partial x_i}+\boldsymbol{\delta}_3\frac{\partial v_3}{\partial x_i}\right)\\
&= -\mu\left[\boldsymbol{\delta}_1\left(\boldsymbol{\delta}_1\frac{\partial v_1}{\partial x_1}+\boldsymbol{\delta}_2\frac{\partial v_2}{\partial x_1}+\boldsymbol{\delta}_3\frac{\partial v_3}{\partial x_1}\right)\right.\\
&\qquad+\boldsymbol{\delta}_2\left(\boldsymbol{\delta}_1\frac{\partial v_1}{\partial x_2}+\boldsymbol{\delta}_2\frac{\partial v_2}{\partial x_2}+\boldsymbol{\delta}_3\frac{\partial v_3}{\partial x_2}\right)\\
&\qquad\left.+\boldsymbol{\delta}_3\left(\boldsymbol{\delta}_1\frac{\partial v_1}{\partial x_3}+\boldsymbol{\delta}_2\frac{\partial v_2}{\partial x_3}+\boldsymbol{\delta}_3\frac{\partial v_3}{\partial x_3}\right)\right]\\
&= -\boldsymbol{\delta}_1\boldsymbol{\delta}_1\mu\frac{\partial v_1}{\partial x_1}-\boldsymbol{\delta}_1\boldsymbol{\delta}_2\mu\frac{\partial v_2}{\partial x_1}-\boldsymbol{\delta}_1\boldsymbol{\delta}_3\mu\frac{\partial v_3}{\partial x_1}\\
&\qquad-\boldsymbol{\delta}_2\boldsymbol{\delta}_1\mu\frac{\partial v_1}{\partial x_2}-\boldsymbol{\delta}_2\boldsymbol{\delta}_2\mu\frac{\partial v_2}{\partial x_2}-\boldsymbol{\delta}_2\boldsymbol{\delta}_3\mu\frac{\partial v_3}{\partial x_2}\\
&\qquad-\boldsymbol{\delta}_3\boldsymbol{\delta}_1\mu\frac{\partial v_1}{\partial x_3}-\boldsymbol{\delta}_3\boldsymbol{\delta}_2\mu\frac{\partial v_2}{\partial x_3}-\boldsymbol{\delta}_3\boldsymbol{\delta}_3\mu\frac{\partial v_3}{\partial x_3}
\end{aligned}$$

$$\begin{aligned}
-\mu(\nabla\mathbf{v})^\dagger &= -\mu\left[\sum_{i=1}^{3}\boldsymbol{\delta}_i\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial v_j}{\partial x_i}\right)\right]^\dagger\\
&= -\mu\sum_{i=1}^{3}\boldsymbol{\delta}_i\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial v_i}{\partial x_j}\right)\\
&= -\mu\sum_{i=1}^{3}\boldsymbol{\delta}_i\left(\boldsymbol{\delta}_1\frac{\partial v_i}{\partial x_1}+\boldsymbol{\delta}_2\frac{\partial v_i}{\partial x_2}+\boldsymbol{\delta}_3\frac{\partial v_i}{\partial x_3}\right)\\
&= -\mu\left[\boldsymbol{\delta}_1\left(\boldsymbol{\delta}_1\frac{\partial v_1}{\partial x_1}+\boldsymbol{\delta}_2\frac{\partial v_1}{\partial x_2}+\boldsymbol{\delta}_3\frac{\partial v_1}{\partial x_3}\right)\right.\\
&\qquad+\boldsymbol{\delta}_2\left(\boldsymbol{\delta}_1\frac{\partial v_2}{\partial x_1}+\boldsymbol{\delta}_2\frac{\partial v_2}{\partial x_2}+\boldsymbol{\delta}_3\frac{\partial v_2}{\partial x_3}\right)\\
&\qquad\left.+\boldsymbol{\delta}_3\left(\boldsymbol{\delta}_1\frac{\partial v_3}{\partial x_1}+\boldsymbol{\delta}_2\frac{\partial v_3}{\partial x_2}+\boldsymbol{\delta}_3\frac{\partial v_3}{\partial x_3}\right)\right]\\
&= -\boldsymbol{\delta}_1\boldsymbol{\delta}_1\mu\frac{\partial v_1}{\partial x_1}-\boldsymbol{\delta}_1\boldsymbol{\delta}_2\mu\frac{\partial v_1}{\partial x_2}-\boldsymbol{\delta}_1\boldsymbol{\delta}_3\mu\frac{\partial v_1}{\partial x_3}\\
&\qquad-\boldsymbol{\delta}_2\boldsymbol{\delta}_1\mu\frac{\partial v_2}{\partial x_1}-\boldsymbol{\delta}_2\boldsymbol{\delta}_2\mu\frac{\partial v_2}{\partial x_2}-\boldsymbol{\delta}_2\boldsymbol{\delta}_3\mu\frac{\partial v_2}{\partial x_3}\\
&\qquad-\boldsymbol{\delta}_3\boldsymbol{\delta}_1\mu\frac{\partial v_3}{\partial x_1}-\boldsymbol{\delta}_3\boldsymbol{\delta}_2\mu\frac{\partial v_3}{\partial x_2}-\boldsymbol{\delta}_3\boldsymbol{\delta}_3\mu\frac{\partial v_3}{\partial x_3}
\end{aligned}$$

$$\begin{aligned}
\frac{2}{3}\mu(\nabla\cdot\mathbf{v})\boldsymbol{\delta} &= \frac{2}{3}\mu\left[\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j v_j\right)\right]\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_l\delta_{kl}\right)\\
&= \frac{2}{3}\mu\left[\sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j)\frac{\partial v_j}{\partial x_i}\right]\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_k\right)\\
&= \frac{2}{3}\mu\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}\frac{\partial v_j}{\partial x_i}\right)\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_k\right)\\
&= \frac{2}{3}\mu\left(\sum_{j=1}^{3}\frac{\partial v_j}{\partial x_j}\right)\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_k\right)\\
&= \frac{2}{3}\mu\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)(\boldsymbol{\delta}_1\boldsymbol{\delta}_1+\boldsymbol{\delta}_2\boldsymbol{\delta}_2+\boldsymbol{\delta}_3\boldsymbol{\delta}_3)\\
&= \boldsymbol{\delta}_1\boldsymbol{\delta}_1\frac{2}{3}\mu\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)\\
&\quad+\boldsymbol{\delta}_2\boldsymbol{\delta}_2\frac{2}{3}\mu\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)\\
&\quad+\boldsymbol{\delta}_3\boldsymbol{\delta}_3\frac{2}{3}\mu\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right).
\end{aligned}$$

Substitute the final forms of each expression into equation (2).

$$\begin{aligned}
\boldsymbol{\delta}_1\boldsymbol{\delta}_1\tau_{11}+\boldsymbol{\delta}_1\boldsymbol{\delta}_2\tau_{12}+\boldsymbol{\delta}_1\boldsymbol{\delta}_3\tau_{13} &= -\boldsymbol{\delta}_1\boldsymbol{\delta}_1\mu\frac{\partial v_1}{\partial x_1}-\boldsymbol{\delta}_1\boldsymbol{\delta}_2\mu\frac{\partial v_2}{\partial x_1}-\boldsymbol{\delta}_1\boldsymbol{\delta}_3\mu\frac{\partial v_3}{\partial x_1}\\
+\boldsymbol{\delta}_2\boldsymbol{\delta}_1\tau_{21}+\boldsymbol{\delta}_2\boldsymbol{\delta}_2\tau_{22}+\boldsymbol{\delta}_2\boldsymbol{\delta}_3\tau_{23} &\quad -\boldsymbol{\delta}_2\boldsymbol{\delta}_1\mu\frac{\partial v_1}{\partial x_2}-\boldsymbol{\delta}_2\boldsymbol{\delta}_2\mu\frac{\partial v_2}{\partial x_2}-\boldsymbol{\delta}_2\boldsymbol{\delta}_3\mu\frac{\partial v_3}{\partial x_2}\\
+\boldsymbol{\delta}_3\boldsymbol{\delta}_1\tau_{31}+\boldsymbol{\delta}_3\boldsymbol{\delta}_2\tau_{32}+\boldsymbol{\delta}_3\boldsymbol{\delta}_3\tau_{33} &\quad -\boldsymbol{\delta}_3\boldsymbol{\delta}_1\mu\frac{\partial v_1}{\partial x_3}-\boldsymbol{\delta}_3\boldsymbol{\delta}_2\mu\frac{\partial v_2}{\partial x_3}-\boldsymbol{\delta}_3\boldsymbol{\delta}_3\mu\frac{\partial v_3}{\partial x_3}\\
&\quad -\boldsymbol{\delta}_1\boldsymbol{\delta}_1\mu\frac{\partial v_1}{\partial x_1}-\boldsymbol{\delta}_1\boldsymbol{\delta}_2\mu\frac{\partial v_1}{\partial x_2}-\boldsymbol{\delta}_1\boldsymbol{\delta}_3\mu\frac{\partial v_1}{\partial x_3}\\
&\quad -\boldsymbol{\delta}_2\boldsymbol{\delta}_1\mu\frac{\partial v_2}{\partial x_1}-\boldsymbol{\delta}_2\boldsymbol{\delta}_2\mu\frac{\partial v_2}{\partial x_2}-\boldsymbol{\delta}_2\boldsymbol{\delta}_3\mu\frac{\partial v_2}{\partial x_3}\\
&\quad -\boldsymbol{\delta}_3\boldsymbol{\delta}_1\mu\frac{\partial v_3}{\partial x_1}-\boldsymbol{\delta}_3\boldsymbol{\delta}_2\mu\frac{\partial v_3}{\partial x_2}-\boldsymbol{\delta}_3\boldsymbol{\delta}_3\mu\frac{\partial v_3}{\partial x_3}\\
&\quad +\boldsymbol{\delta}_1\boldsymbol{\delta}_1\frac{2}{3}\mu\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)\\
&\quad +\boldsymbol{\delta}_2\boldsymbol{\delta}_2\frac{2}{3}\mu\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)\\
&\quad +\boldsymbol{\delta}_3\boldsymbol{\delta}_3\frac{2}{3}\mu\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)
\end{aligned}$$

The component of $\boldsymbol{\delta}_1\boldsymbol{\delta}_1$ on the left side must be equal to the component of $\boldsymbol{\delta}_1\boldsymbol{\delta}_1$ on the right side, and the same is true for all the other components. Therefore, equation (2) is equivalent to the following nine equations in Cartesian coordinates.

$$\begin{aligned}
\tau_{11} &= -2\mu\frac{\partial v_1}{\partial x_1} + \frac{2}{3}\mu\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right) \\
\tau_{12} &= -\mu\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) \\
\tau_{13} &= -\mu\left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}\right) \\
\tau_{21} &= -\mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) \\
\tau_{22} &= -2\mu\frac{\partial v_2}{\partial x_2} + \frac{2}{3}\mu\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right) \\
\tau_{23} &= -\mu\left(\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}\right) \\
\tau_{31} &= -\mu\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right) \\
\tau_{32} &= -\mu\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right) \\
\tau_{33} &= -2\mu\frac{\partial v_3}{\partial x_3} + \frac{2}{3}\mu\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right)
\end{aligned}$$

Do note that $\tau_{12} = \tau_{21}$, $\tau_{13} = \tau_{31}$, and $\tau_{23} = \tau_{32}$.