Exercise 8

Develop an alternative expression for $[\nabla \times [\nabla \cdot s\mathbf{vv}]]$.

Solution

$$
\begin{aligned}
\nabla \times [\nabla \cdot s\mathbf{vv}] &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \cdot s \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k v_k\right) \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l v_l\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \cdot s \left(\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k \boldsymbol{\delta}_l v_k v_l\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k) \boldsymbol{\delta}_l \frac{\partial}{\partial x_j}(s v_k v_l)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \delta_{jk} \boldsymbol{\delta}_l \frac{\partial}{\partial x_j}(s v_k v_l)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \frac{\partial}{\partial x_k}(s v_k v_l)\right] \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_l) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k}(s v_k v_l) \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k}(s v_k v_l)
\end{aligned}
$$

Because of ε_{ilm}, there's going to be a cross product, so isolate the sums that contain the i, l, and m indices.

$$
\begin{aligned}
&= \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \left[\sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial}{\partial x_i}(s v_k v_l)\right] \\
&= \sum_{k=1}^{3} \frac{\partial}{\partial x_k} [\nabla \times s v_k \mathbf{v}]
\end{aligned}
$$

In order to bring the k-index out, use the transpose.

$$
\begin{aligned}
&= \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \{\nabla \times s\mathbf{vv}\}_k^{\dagger} \\
&= \nabla \cdot \{\nabla \times s\mathbf{vv}\}^{\dagger}
\end{aligned}
$$

Therefore,

$$
\nabla \times [\nabla \cdot s\mathbf{vv}] = \nabla \cdot \{\nabla \times s\mathbf{vv}\}^{\dagger}.
$$