Exercise 9

If $\mathbf{r}$ is the position vector and r is its magnitude, verify that

$$\textbf{(a)}\quad \nabla\frac{1}{r} = -\frac{\mathbf{r}}{r^3} \qquad \textbf{(c)}\quad \nabla(\mathbf{a}\cdot\mathbf{r}) = \mathbf{a} \qquad \text{if } \mathbf{a} \text{ is a constant vector}$$

$$\textbf{(b)}\quad \nabla f(r) = \frac{1}{r}\frac{df}{dr}\mathbf{r}$$

Solution

Part (a)

$$\begin{aligned}
\nabla\frac{1}{r} &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\left(\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\right) = \left(\boldsymbol{\delta}_1\frac{\partial}{\partial x_1}+\boldsymbol{\delta}_2\frac{\partial}{\partial x_2}+\boldsymbol{\delta}_3\frac{\partial}{\partial x_3}\right)\left(\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\right)\\
&= \boldsymbol{\delta}_1\frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\right)+\boldsymbol{\delta}_2\frac{\partial}{\partial x_2}\left(\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\right)+\boldsymbol{\delta}_3\frac{\partial}{\partial x_3}\left(\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\right)\\
&= \boldsymbol{\delta}_1\left[\frac{-x_1}{(x_1^2+x_2^2+x_3^2)^{3/2}}\right]+\boldsymbol{\delta}_2\left[\frac{-x_2}{(x_1^2+x_2^2+x_3^2)^{3/2}}\right]+\boldsymbol{\delta}_3\left[\frac{-x_3}{(x_1^2+x_2^2+x_3^2)^{3/2}}\right]\\
&= -\frac{1}{(x_1^2+x_2^2+x_3^2)^{3/2}}(\boldsymbol{\delta}_1x_1+\boldsymbol{\delta}_2x_2+\boldsymbol{\delta}_3x_3) = -\frac{1}{(\sqrt{x_1^2+x_2^2+x_3^2})^3}(\boldsymbol{\delta}_1x_1+\boldsymbol{\delta}_2x_2+\boldsymbol{\delta}_3x_3)\\
&= -\frac{1}{r^3}\mathbf{r}
\end{aligned}$$

Part (b)

$$\nabla f(r) = \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)f(r) = \left(\boldsymbol{\delta}_1\frac{\partial}{\partial x_1}+\boldsymbol{\delta}_2\frac{\partial}{\partial x_2}+\boldsymbol{\delta}_3\frac{\partial}{\partial x_3}\right)f(r) = \boldsymbol{\delta}_1\frac{\partial f}{\partial x_1}+\boldsymbol{\delta}_2\frac{\partial f}{\partial x_2}+\boldsymbol{\delta}_3\frac{\partial f}{\partial x_3}$$

f is solely a function of r, not x_1, x_2, or x_3, so we have to use the chain rule.

$$\begin{aligned}
&= \boldsymbol{\delta}_1\frac{df}{dr}\frac{\partial r}{\partial x_1}+\boldsymbol{\delta}_2\frac{df}{dr}\frac{\partial r}{\partial x_2}+\boldsymbol{\delta}_3\frac{df}{dr}\frac{\partial r}{\partial x_3}\\
&= \boldsymbol{\delta}_1\frac{df}{dr}\frac{\partial}{\partial x_1}\left(\sqrt{x_1^2+x_2^2+x_3^2}\right)+\boldsymbol{\delta}_2\frac{df}{dr}\frac{\partial}{\partial x_2}\left(\sqrt{x_1^2+x_2^2+x_3^2}\right)+\boldsymbol{\delta}_3\frac{df}{dr}\frac{\partial}{\partial x_3}\left(\sqrt{x_1^2+x_2^2+x_3^2}\right)\\
&= \boldsymbol{\delta}_1\frac{df}{dr}\left(\frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}}\right)+\boldsymbol{\delta}_2\frac{df}{dr}\left(\frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}}\right)+\boldsymbol{\delta}_3\frac{df}{dr}\left(\frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}\right)\\
&= \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\frac{df}{dr}(\boldsymbol{\delta}_1x_1+\boldsymbol{\delta}_2x_2+\boldsymbol{\delta}_3x_3)\\
&= \frac{1}{r}\frac{df}{dr}\mathbf{r}
\end{aligned}$$

Part (c)

$$\begin{aligned}
\nabla(\mathbf{a}\cdot\mathbf{r}) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j a_j\right)\cdot\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k x_k\right)\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\left[\sum_{j=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)a_j x_k\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\delta_{jk}a_j x_k\right) \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{3}a_j x_j\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}(a_j x_j) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_i a_j\frac{\partial x_j}{\partial x_i} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_i a_j\delta_{ji} \\
&= \sum_{i=1}^{3}\boldsymbol{\delta}_i a_i \\
&= \mathbf{a}
\end{aligned}$$