Exercise 2

A field $\mathbf{v}(x, y, z)$ is said to be *irrotational* if $[\nabla \times \mathbf{v}] = 0$. Which of the following fields are irrotational?

(a) $v_x = by \quad v_y = 0 \quad v_z = 0$
(b) $v_x = bx \quad v_y = 0 \quad v_z = 0$
(c) $v_x = by \quad v_y = bx \quad v_z = 0$
(d) $v_x = -by \quad v_y = bx \quad v_z = 0$

Solution

$\nabla \times \mathbf{v}$ is the curl of the vector field $\mathbf{v}$, and it is evaluated with the following determinant.

$$\nabla \times \mathbf{v} = \begin{vmatrix} \boldsymbol{\delta}_1 & \boldsymbol{\delta}_2 & \boldsymbol{\delta}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$\textbf{(a)} \quad \nabla \times \mathbf{v} = \begin{vmatrix} \boldsymbol{\delta}_1 & \boldsymbol{\delta}_2 & \boldsymbol{\delta}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ by & 0 & 0 \end{vmatrix} = 0\boldsymbol{\delta}_1 - \left[0 - \frac{\partial}{\partial z}(by)\right]\boldsymbol{\delta}_2 + \left[0 - \frac{\partial}{\partial y}(by)\right]\boldsymbol{\delta}_3 = -b\boldsymbol{\delta}_3$$

$$\textbf{(b)} \quad \nabla \times \mathbf{v} = \begin{vmatrix} \boldsymbol{\delta}_1 & \boldsymbol{\delta}_2 & \boldsymbol{\delta}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bx & 0 & 0 \end{vmatrix} = 0\boldsymbol{\delta}_1 - \left[0 - \frac{\partial}{\partial z}(bx)\right]\boldsymbol{\delta}_2 + \left[0 - \frac{\partial}{\partial y}(bx)\right]\boldsymbol{\delta}_3 = \mathbf{0}$$

$$\textbf{(c)} \quad \nabla \times \mathbf{v} = \begin{vmatrix} \boldsymbol{\delta}_1 & \boldsymbol{\delta}_2 & \boldsymbol{\delta}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ by & bx & 0 \end{vmatrix} = \left[0 - \frac{\partial}{\partial z}(bx)\right]\boldsymbol{\delta}_1 - \left[0 - \frac{\partial}{\partial z}(by)\right]\boldsymbol{\delta}_2 + \left[\frac{\partial}{\partial x}(bx) - \frac{\partial}{\partial y}(by)\right]\boldsymbol{\delta}_3 = \mathbf{0}$$

$$\textbf{(d)} \quad \nabla \times \mathbf{v} = \begin{vmatrix} \boldsymbol{\delta}_1 & \boldsymbol{\delta}_2 & \boldsymbol{\delta}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -by & bx & 0 \end{vmatrix} = \left[0 - \frac{\partial}{\partial z}(bx)\right]\boldsymbol{\delta}_1 - \left[0 - \frac{\partial}{\partial z}(-by)\right]\boldsymbol{\delta}_2 + \left[\frac{\partial}{\partial x}(bx) - \frac{\partial}{\partial y}(-by)\right]\boldsymbol{\delta}_3$$
$$= 2b\boldsymbol{\delta}_3$$

Therefore, the vector fields in **(b)** and **(c)** are irrotational.