Exercise 7

If $\mathbf{r}$ is the position vector (with components x_1, x_2, x_3) and $\mathbf{v}$ is any vector, show that

(a) $(\nabla \cdot \mathbf{r}) = 3$

(b) $[\nabla \times \mathbf{r}] = 0$

(c) $[\mathbf{r} \times [\nabla \cdot \mathbf{vv}]] = [\nabla \cdot \mathbf{v}[\mathbf{r} \times \mathbf{v}]]$ (where $\mathbf{v}$ is a function of position)

[TYPO: 0 should be in bold, as the curl operator yields a vector, not a scalar.]

Solution

Part (a)

$$\nabla \cdot \mathbf{r} = \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j x_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial x_j}{\partial x_i} = \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j)\delta_{ji} = \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}\delta_{ji}$$

$$= \sum_{i=1}^{3} \delta_{ii} = \sum_{i=1}^{3} 1 = 1 + 1 + 1 = 3$$

Part (b)

$$\nabla \times \mathbf{r} = \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j x_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) \frac{\partial x_j}{\partial x_i} = \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j)\delta_{ji}$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \delta_{ji} = \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{jjk} = \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k (0) = \mathbf{0}$$

Part (c)

$$
\begin{aligned}
\mathbf{r} \times [\nabla \cdot \mathbf{vv}] &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i x_i\right) \times \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \cdot \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k v_k\right)\left(\sum_{l=1}^{3} \boldsymbol{\delta}_l v_l\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i x_i\right) \times \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \cdot \left(\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k \boldsymbol{\delta}_l v_k v_l\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i x_i\right) \times \left[\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k)\boldsymbol{\delta}_l \frac{\partial}{\partial x_j}(v_k v_l)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i x_i\right) \times \left[\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \delta_{jk}\boldsymbol{\delta}_l \frac{\partial}{\partial x_j}(v_k v_l)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i x_i\right) \times \left[\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \frac{\partial}{\partial x_k}(v_k v_l)\right] \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_l) x_i \frac{\partial}{\partial x_k}(v_k v_l) \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} x_i \frac{\partial}{\partial x_k}(v_k v_l) \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \left[\frac{\partial}{\partial x_k}(x_i v_k v_l) - v_k v_l \frac{\partial x_i}{\partial x_k}\right] \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \left[\frac{\partial}{\partial x_k}(x_i v_k v_l) - v_k v_l \delta_{ik}\right] \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial}{\partial x_k}(x_i v_k v_l) - \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} v_k v_l \delta_{ik} \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial}{\partial x_k}(x_i v_k v_l) - \sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{klm} v_k v_l \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial}{\partial x_k}(x_i v_k v_l) - [\mathbf{v} \times \mathbf{v}] \\
&= \sum_{k=1}^{3} \frac{\partial}{\partial x_k} v_k \left(\sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} x_i v_l\right) - \mathbf{0} \\
&= \sum_{k=1}^{3} \frac{\partial}{\partial x_k} v_k [\mathbf{r} \times \mathbf{v}] \\
&= \nabla \cdot \mathbf{v}[\mathbf{r} \times \mathbf{v}]
\end{aligned}
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