Exercise 2

Use the same vector field to evaluate both sides of Eq. A.5-4 for the face $x_1 = 1$ in Exercise 1.

Solution

Eq. A.5-4 states Stokes's theorem,

$$\iint_S (\hat{\mathbf{n}} \cdot [\nabla \times \mathbf{v}])\, dS = \oint_C (\hat{\mathbf{t}} \cdot \mathbf{v})\, dC,$$

where $\hat{\mathbf{t}}$ is a unit vector tangent to the integration path C and $\hat{\mathbf{n}}$ is a unit vector in the direction the thumb points when the fingers of the right hand curl in the direction of the path. From Exercise 1, we have

$$\mathbf{v} = \boldsymbol{\delta}_1 x_1 + \boldsymbol{\delta}_2 x_3 + \boldsymbol{\delta}_3 x_2.$$

The Left-hand Side

To evaluate the left-hand side, determine the curl of $\mathbf{v}$.

$$\nabla \times \mathbf{v} = \begin{vmatrix} \boldsymbol{\delta}_1 & \boldsymbol{\delta}_2 & \boldsymbol{\delta}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ x_1 & x_3 & x_2 \end{vmatrix} = \mathbf{0}$$

As a result, the left-hand side is zero.

$$\begin{aligned} \iint_S (\hat{\mathbf{n}} \cdot [\nabla \times \mathbf{v}])\, dS &= \iint_S (\hat{\mathbf{n}} \cdot \mathbf{0})\, dS \\ &= 0 \end{aligned}$$

The Right-hand Side

The boundary of the $x_1 = 1$ face is made up of four paths as shown in the figure, so the closed loop integral splits up into four single integrals.

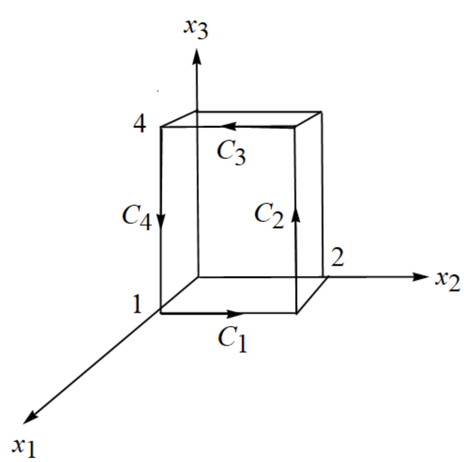

Figure 1: Schematic of the $x_1 = 1$ face and the integration path around its boundary.

$$\oint_C (\hat{\mathbf{t}} \cdot \mathbf{v})\, dC = \int_{C_1} (\hat{\mathbf{t}} \cdot \mathbf{v})\, dC + \int_{C_2} (\hat{\mathbf{t}} \cdot \mathbf{v})\, dC + \int_{C_3} (\hat{\mathbf{t}} \cdot \mathbf{v})\, dC + \int_{C_4} (\hat{\mathbf{t}} \cdot \mathbf{v})\, dC$$

The tangent vector along C_1 is $\boldsymbol{\delta}_2$, the tangent vector along C_2 is $\boldsymbol{\delta}_3$, the tangent vector along C_3 is $-\boldsymbol{\delta}_2$, and the tangent vector along C_4 is $-\boldsymbol{\delta}_3$.

$$\begin{aligned}\oint_C (\hat{\mathbf{t}} \cdot \mathbf{v})\, dC = \int_{C_1} \boldsymbol{\delta}_2 \cdot (\boldsymbol{\delta}_1 x_1 + \boldsymbol{\delta}_2 x_3 + \boldsymbol{\delta}_3 x_2)\, dC + \int_{C_2} \boldsymbol{\delta}_3 \cdot (\boldsymbol{\delta}_1 x_1 + \boldsymbol{\delta}_2 x_3 + \boldsymbol{\delta}_3 x_2)\, dC \\ + \int_{C_3} (-\boldsymbol{\delta}_2) \cdot (\boldsymbol{\delta}_1 x_1 + \boldsymbol{\delta}_2 x_3 + \boldsymbol{\delta}_3 x_2)\, dC + \int_{C_4} (-\boldsymbol{\delta}_3) \cdot (\boldsymbol{\delta}_1 x_1 + \boldsymbol{\delta}_2 x_3 + \boldsymbol{\delta}_3 x_2)\, dC\end{aligned}$$

Evaluate the dot products.

$$\oint_C (\hat{\mathbf{t}} \cdot \mathbf{v})\, dC = \int_{C_1} x_3\, dC + \int_{C_2} x_2\, dC + \int_{C_3} (-x_3)\, dC + \int_{C_4} (-x_2)\, dC$$

Along the C_1 path, $x_3 = 0$; along the C_2 path, $x_2 = 2$; along the C_3 path, $x_3 = 4$; and along the C_4 path, $x_2 = 0$.

$$\oint_C (\hat{\mathbf{t}} \cdot \mathbf{v})\, dC = \int_{C_1} 0\, dC + \int_{C_2} 2\, dC + \int_{C_3} (-4)\, dC + \int_{C_4} (-0)\, dC$$

Bring the constants in front of the nonzero integrals.

$$\oint_C (\hat{\mathbf{t}} \cdot \mathbf{v})\, dC = 2 \int_{C_2} dC - 4 \int_{C_3} dC$$

The length of the C_2 path is 4, and the length of the C_3 path is 2. Therefore,

$$\begin{aligned}\oint_C (\hat{\mathbf{t}} \cdot \mathbf{v})\, dC &= 2(4) - 4(2) \\ &= 0.\end{aligned}$$

We conclude that Stokes's theorem is verified.