Exercise 3

Consider the time-dependent scalar function:

\[ s = x + y + zt \]

Evaluate both sides of Eq. A.5-5 over the volume bounded by the planes: \( x = 0, \) \( x = t; \) \( y = 0, \) \( y = 2t; \) \( z = 0, \) \( z = 4t. \) The quantities \( x, y, z, t \) are dimensionless.

Solution

Eq. A.5-5 is the Leibniz formula for differentiating a volume integral,

\[
\frac{d}{dt} \iiint_{V} s \, dV = \iiint_{V} \frac{\partial s}{\partial t} \, dV + \iint_{S} s (\mathbf{v}_S \cdot \hat{n}) \, dS,
\]

where \( V \) is a closed volume whose surface elements are moving at velocity \( \mathbf{v}_S. \)

**The Left-hand Side**

\[
\frac{d}{dt} \iiint_{V} s \, dV = \frac{d}{dt} \iiint_{V} (x + y + zt) \, dV
\]

\[
= \frac{d}{dt} \int_{0}^{4t} \int_{0}^{2t} \int_{0}^{t} (x + y + zt) \, dx \, dy \, dz
\]

\[
= \frac{d}{dt} \int_{0}^{4t} \int_{0}^{2t} \left( \frac{x^2}{2} + xy + xzt \right)_{0}^{t} \, dy \, dz
\]

\[
= \frac{d}{dt} \int_{0}^{4t} \int_{0}^{2t} \left( \frac{t^2}{2} + ty + t^2z \right) \, dy \, dz
\]

\[
= \frac{d}{dt} \int_{0}^{4t} \left( \frac{t^2}{2} y + \frac{y^2}{2} + t^2zy \right)_{0}^{2t} \, dz
\]

\[
= \frac{d}{dt} \int_{0}^{4t} (3t^3 + 2t^3z) \, dz
\]

\[
= \frac{d}{dt} \left( 3t^3z + 2t^3 \frac{z^2}{2} \right)_{0}^{4t}
\]

\[
= \frac{d}{dt} \left( 12t^4 + 16t^5 \right)
\]

\[
= 48t^3 + 80t^4
\]

\[
= 16t^3(3 + 5t)
\]
The Right-hand Side

Evaluate the first term on the right-hand side.
\[
\iiint_V \frac{\partial s}{\partial t} \, dV = \iiint_V \frac{\partial}{\partial t} (x + y + zt) \, dV
\]
\[
= \iiint_V z \, dV
\]
\[
= \int_0^4 \int_0^{2t} \int_0^t z \, dx \, dy \, dz
\]
\[
= \left( \int_0^t dx \right) \left( \int_0^{2t} dy \right) \left( \int_0^t z \, dz \right)
\]
\[
= (t - 0)(2t - 0) \left( \frac{z^2}{2} \right) \bigg|_0^t
\]
\[
= (t)(2t)(8t^2)
\]
\[
= 16t^4
\]

Now the second term will be evaluated. The volume \( V \) is a rectangular box where one of its six faces expands in the \( x \)-direction with speed \( dx/dt = 1 \), another face expands in the \( y \)-direction with speed \( dy/dt = 2 \), and another face expands in the \( z \)-direction with speed \( dz/dt = 4 \). The faces at \( x = 0 \), \( y = 0 \), and \( z = 0 \) do not move, so they will not contribute to the surface integral. The closed surface integral will thus have three nonzero terms.
\[
\iint_S (\mathbf{v}_S \cdot \hat{n}) \, dS = \iint_{S_1} (\mathbf{v}_S \cdot \hat{n}) \, dS + \iint_{S_2} (\mathbf{v}_S \cdot \hat{n}) \, dS + \iint_{S_3} (\mathbf{v}_S \cdot \hat{n}) \, dS
\]
Each of the three faces moves in the direction perpendicular to itself, so \( (\mathbf{v}_S \cdot \hat{n}) \) is just the speed that each face moves.
\[
\iint_S (\mathbf{v}_S \cdot \hat{n}) \, dS = \iint_{S_1} s(1) \, dS + \iint_{S_2} s(2) \, dS + \iint_{S_3} s(4) \, dS
\]
Bring the constants in front.
\[
\iint_S (\mathbf{v}_S \cdot \hat{n}) \, dS = \iint_{S_1} s \, dS + 2 \iint_{S_2} s \, dS + 4 \iint_{S_3} s \, dS
\]
Our task now is to evaluate the double integral of \( s \) over the moving faces. The face moving in the \( x \)-direction will be in \( dy \) and \( dz \), the one moving in the \( y \)-direction will be in \( dx \) and \( dz \), and the one moving in the \( z \)-direction will be in \( dx \) and \( dy \).
\[
\iiint_S (\mathbf{v}_S \cdot \hat{n}) \, dS = \int_0^t \int_0^{2t} s \, dy \, dz + 2 \int_0^t \int_0^{2t} s \, dx \, dz + 4 \int_0^t \int_0^t s \, dx \, dy
\]
\[
= \int_0^t \int_0^{2t} (x + y + zt) \, dy \, dz + 2 \int_0^t \int_0^t (x + y + zt) \, dx \, dz + 4 \int_0^t \int_0^t (x + y + zt) \, dx \, dy
\]
Plug in \( x = t \) into the first double integral, \( y = 2t \) into the second double integral, and \( z = 4t \) into the third double integral.
\[
= \int_0^t \int_0^{2t} (t + y + zt) \, dy \, dz + 2 \int_0^t \int_0^t (x + 2t + zt) \, dx \, dz + 4 \int_0^t \int_0^t (x + y + 4t^2) \, dx \, dy
\]
\[
= \int_0^t \left( ty + \frac{y^2}{2} + yzt \right) \bigg|_0^{2t} \, dz + 2 \int_0^t \left( \frac{x^2}{2} + 2tx + xzt \right) \bigg|_0^t \, dz + 4 \int_0^t \left( \frac{x^2}{2} + xy + 4t^2x \right) \bigg|_0^t \, dy
\]
Therefore,

\[
\iiint_V \frac{\partial s}{\partial t} \, dV + \iiint_S s(\mathbf{v}_S \cdot \mathbf{n}) \, dS = 16t^4 + 48t^3 + 64t^4
\]

\[
= 48t^4 + 80t^4
\]

\[
= 16t^3(3 + 5t).
\]

We conclude that the Leibniz formula is verified.