Exercise 1

Consider the vector field

$$\mathbf{v} = \boldsymbol{\delta}_1 x_1 + \boldsymbol{\delta}_2 x_3 + \boldsymbol{\delta}_3 x_2$$

Evaluate both sides of Eq. A.5-1 over the region bounded by the planes $x_1 = 0$, $x_1 = 1$; $x_2 = 0$, $x_2 = 2$; $x_3 = 0$, $x_3 = 4$.

Solution

Eq. A.5-1 states the divergence theorem,

$$\iiint_V (\nabla \cdot \mathbf{v})\, dV = \oiint_S (\hat{\mathbf{n}} \cdot \mathbf{v})\, dS,$$

where V is a closed region in space enclosed by a surface S and $\hat{\mathbf{n}}$ is the unit vector normal to this surface directed outwardly.

The Left-hand Side

To evaluate the left-hand side, determine the divergence of $\mathbf{v}$.

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x_1}(x_1) + \frac{\partial}{\partial x_2}(x_3) + \frac{\partial}{\partial x_3}(x_2) = 1$$

The left-hand side simplifies to the volume of the region in question.

$$\begin{aligned}
\iiint_V (\nabla \cdot \mathbf{v})\, dV = \iiint_V dV &= \int_0^4 \int_0^2 \int_0^1 dx_1\, dx_2\, dx_3 \\
&= \left(\int_0^1 dx_1\right)\left(\int_0^2 dx_2\right)\left(\int_0^4 dx_3\right) \\
&= (1-0)(2-0)(4-0) \\
&= 8
\end{aligned}$$

The Right-hand Side

The volume is a rectangular box with six faces, so the closed surface integral splits up into six double integrals—one for each face.

$$\begin{aligned}
\oiint_S (\hat{\mathbf{n}} \cdot \mathbf{v})\, dS = \iint_{S_1} (\hat{\mathbf{n}} \cdot \mathbf{v})\, dS + \iint_{S_2} (\hat{\mathbf{n}} \cdot \mathbf{v})\, dS + \iint_{S_3} (\hat{\mathbf{n}} \cdot \mathbf{v})\, dS \\
+ \iint_{S_4} (\hat{\mathbf{n}} \cdot \mathbf{v})\, dS + \iint_{S_5} (\hat{\mathbf{n}} \cdot \mathbf{v})\, dS + \iint_{S_6} (\hat{\mathbf{n}} \cdot \mathbf{v})\, dS
\end{aligned}$$

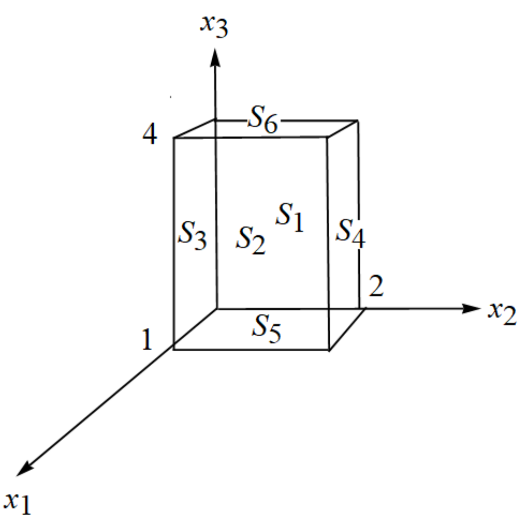

Figure 1: Schematic of the rectangular box and its six faces.

$\boldsymbol{\delta}_1$ is the unit vector normal to the two faces at $x_1 = 0$ and $x_1 = 1$, $\boldsymbol{\delta}_2$ is the unit vector normal to the two faces at $x_2 = 0$ and $x_2 = 2$, and $\boldsymbol{\delta}_3$ is the unit vector normal to the two faces at $x_3 = 0$ and $x_3 = 4$. Since the outward normal vectors at $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ point in the negative direction, there's a minus sign in front of these unit vectors.

$$\begin{aligned}\oiint_S (\hat{\mathbf{n}} \cdot \mathbf{v})\, dS &= \iint_{S_1} (-\boldsymbol{\delta}_1) \cdot (\boldsymbol{\delta}_1 x_1 + \boldsymbol{\delta}_2 x_3 + \boldsymbol{\delta}_3 x_2)\, dS + \iint_{S_2} \boldsymbol{\delta}_1 \cdot (\boldsymbol{\delta}_1 x_1 + \boldsymbol{\delta}_2 x_3 + \boldsymbol{\delta}_3 x_2)\, dS \\ &\quad + \iint_{S_3} (-\boldsymbol{\delta}_2) \cdot (\boldsymbol{\delta}_1 x_1 + \boldsymbol{\delta}_2 x_3 + \boldsymbol{\delta}_3 x_2)\, dS + \iint_{S_4} \boldsymbol{\delta}_2 \cdot (\boldsymbol{\delta}_1 x_1 + \boldsymbol{\delta}_2 x_3 + \boldsymbol{\delta}_3 x_2)\, dS \\ &\quad + \iint_{S_5} (-\boldsymbol{\delta}_3) \cdot (\boldsymbol{\delta}_1 x_1 + \boldsymbol{\delta}_2 x_3 + \boldsymbol{\delta}_3 x_2)\, dS + \iint_{S_6} \boldsymbol{\delta}_3 \cdot (\boldsymbol{\delta}_1 x_1 + \boldsymbol{\delta}_2 x_3 + \boldsymbol{\delta}_3 x_2)\, dS\end{aligned}$$

Evaluate the dot products.

$$\begin{aligned}\oiint_S (\hat{\mathbf{n}} \cdot \mathbf{v})\, dS &= \iint_{S_1} (-x_1)\, dS + \iint_{S_2} x_1\, dS + \iint_{S_3} (-x_3)\, dS \\ &\quad + \iint_{S_4} x_3\, dS + \iint_{S_5} (-x_2)\, dS + \iint_{S_6} x_2\, dS\end{aligned}$$

At S_1, $x_1 = 0$; at S_2, $x_1 = 1$; at S_3, $x_2 = 0$; at S_4, $x_2 = 2$; at S_5, $x_3 = 0$; and at S_6, $x_3 = 4$.

$$\begin{aligned}\oiint_S (\hat{\mathbf{n}} \cdot \mathbf{v})\, dS &= \int_0^4 \int_0^2 (-0)\, dx_2\, dx_3 + \int_0^4 \int_0^2 1\, dx_2\, dx_3 + \int_0^4 \int_0^1 (-x_3)\, dx_1\, dx_3 \\ &\quad + \int_0^4 \int_0^1 x_3\, dx_1\, dx_3 + \int_0^2 \int_0^1 (-x_2)\, dx_1\, dx_2 + \int_0^2 \int_0^1 x_2\, dx_1\, dx_2\end{aligned}$$

The first double integral is zero. Bringing the constants in front of the others, we find that four of them cancel.

$$\begin{aligned}\oiint_S (\hat{\mathbf{n}} \cdot \mathbf{v})\, dS &= \int_0^4 \int_0^2 dx_2\, dx_3 - \cancel{\int_0^4 \int_0^1 x_3\, dx_1\, dx_3} + \cancel{\int_0^4 \int_0^1 x_3\, dx_1\, dx_3} \\ &\quad - \cancel{\int_0^2 \int_0^1 x_2\, dx_1\, dx_2} + \cancel{\int_0^2 \int_0^1 x_2\, dx_1\, dx_2}\end{aligned}$$

Therefore,

$$\begin{aligned}\oiint_S (\hat{\mathbf{n}} \cdot \mathbf{v})\, dS &= \int_0^4 \int_0^2 dx_2\, dx_3 \\ &= \left(\int_0^2 dx_2\right)\left(\int_0^4 dx_3\right) \\ &= (2-0)(4-0) \\ &= 8.\end{aligned}$$

We conclude that the divergence theorem is verified.