Exercise 3

Consider the time-dependent scalar function:

$$s = x + y + zt$$

Evaluate both sides of Eq. A.5-5 over the volume bounded by the planes: $x = 0$, $x = t$; $y = 0$, $y = 2t$; $z = 0$, $z = 4t$. The quantities x, y, z, t are dimensionless.

Solution

Eq. A.5-5 is the Leibniz formula for differentiating a volume integral,

$$\frac{d}{dt}\iiint_V s\,dV = \iiint_V \frac{\partial s}{\partial t}\,dV + \oiint_S s(\mathbf{v}_S \cdot \hat{\mathbf{n}})\,dS,$$

where V is a closed volume whose surface elements are moving at velocity $\mathbf{v}_S$.

The Left-hand Side

$$\begin{aligned}
\frac{d}{dt}\iiint_V s\,dV &= \frac{d}{dt}\iiint_V (x + y + zt)\,dV \\
&= \frac{d}{dt}\int_0^{4t}\int_0^{2t}\int_0^{t} (x + y + zt)\,dx\,dy\,dz \\
&= \frac{d}{dt}\int_0^{4t}\int_0^{2t} \left(\frac{x^2}{2} + xy + xzt\right)\Bigg|_0^t\,dy\,dz \\
&= \frac{d}{dt}\int_0^{4t}\int_0^{2t} \left(\frac{t^2}{2} + ty + t^2 z\right)\,dy\,dz \\
&= \frac{d}{dt}\int_0^{4t} \left(\frac{t^2}{2}y + t\frac{y^2}{2} + t^2 zy\right)\Bigg|_0^{2t}\,dz \\
&= \frac{d}{dt}\int_0^{4t} (3t^3 + 2t^3 z)\,dz \\
&= \frac{d}{dt}\left(3t^3 z + 2t^3\frac{z^2}{2}\right)\Bigg|_0^{4t} \\
&= \frac{d}{dt}(12t^4 + 16t^5) \\
&= 48t^3 + 80t^4 \\
&= 16t^3(3 + 5t)
\end{aligned}$$

The Right-hand Side

Evaluate the first term on the right-hand side.

$$\begin{aligned}\iiint_V \frac{\partial s}{\partial t}\,dV &= \iiint_V \frac{\partial}{\partial t}(x+y+zt)\,dV \\ &= \iiint_V z\,dV \\ &= \int_0^{4t}\int_0^{2t}\int_0^{t} z\,dx\,dy\,dz \\ &= \left(\int_0^t dx\right)\left(\int_0^{2t} dy\right)\left(\int_0^{4t} z\,dz\right) \\ &= (t-0)(2t-0)\left(\frac{z^2}{2}\right)\Big|_0^{4t} \\ &= (t)(2t)(8t^2) \\ &= 16t^4\end{aligned}$$

Now the second term will be evaluated. The volume V is a rectangular box where one of its six faces expands in the x-direction with speed $dx/dt = 1$, another face expands in the y-direction with speed $dy/dt = 2$, and another face expands in the z-direction with speed $dz/dt = 4$. The faces at $x = 0$, $y = 0$, and $z = 0$ do not move, so they will not contribute to the surface integral. The closed surface integral will thus have three nonzero terms.

$$\oiint_S s(\mathbf{v}_S\cdot\hat{\mathbf{n}})\,dS = \iint_{S_1} s(\mathbf{v}_S\cdot\hat{\mathbf{n}})\,dS + \iint_{S_2} s(\mathbf{v}_S\cdot\hat{\mathbf{n}})\,dS + \iint_{S_3} s(\mathbf{v}_S\cdot\hat{\mathbf{n}})\,dS$$

Each of the three faces moves in the direction perpendicular to itself, so $(\mathbf{v}_S\cdot\hat{\mathbf{n}})$ is just the speed that each face moves.

$$\oiint_S s(\mathbf{v}_S\cdot\hat{\mathbf{n}})\,dS = \iint_{S_1} s(1)\,dS + \iint_{S_2} s(2)\,dS + \iint_{S_3} s(4)\,dS$$

Bring the constants in front.

$$\oiint_S s(\mathbf{v}_S\cdot\hat{\mathbf{n}})\,dS = \iint_{S_1} s\,dS + 2\iint_{S_2} s\,dS + 4\iint_{S_3} s\,dS$$

Our task now is to evaluate the double integral of s over the moving faces. The face moving in the x-direction will be in dy and dz, the one moving in the y-direction will be in dx and dz, and the one moving in the z-direction will be in dx and dy.

$$\begin{aligned}\oiint_S s(\mathbf{v}_S\cdot\hat{\mathbf{n}})\,dS &= \int_0^{4t}\int_0^{2t} s\,dy\,dz + 2\int_0^{4t}\int_0^{t} s\,dx\,dz + 4\int_0^{2t}\int_0^{t} s\,dx\,dy \\ &= \int_0^{4t}\int_0^{2t}(x+y+zt)\,dy\,dz + 2\int_0^{4t}\int_0^{t}(x+y+zt)\,dx\,dz + 4\int_0^{2t}\int_0^{t}(x+y+zt)\,dx\,dy\end{aligned}$$

Plug in $x = t$ into the first double integral, $y = 2t$ into the second double integral, and $z = 4t$ into the third double integral.

$$\begin{aligned}&= \int_0^{4t}\int_0^{2t}(t+y+zt)\,dy\,dz + 2\int_0^{4t}\int_0^{t}(x+2t+zt)\,dx\,dz + 4\int_0^{2t}\int_0^{t}(x+y+4t^2)\,dx\,dy \\ &= \int_0^{4t}\left(ty+\frac{y^2}{2}+yzt\right)\Big|_0^{2t} dz + 2\int_0^{4t}\left(\frac{x^2}{2}+2tx+xzt\right)\Big|_0^{t} dz + 4\int_0^{2t}\left(\frac{x^2}{2}+xy+4t^2x\right)\Big|_0^{t} dy\end{aligned}$$

$$\begin{aligned}\oiint_S s(\mathbf{v}_S \cdot \hat{\mathbf{n}})\, dS &= \int_0^{4t} (4t^2 + 2t^2 z)\, dz + 2\int_0^{4t} \left(\frac{5t^2}{2} + t^2 z\right)\, dz + 4\int_0^{2t} \left(\frac{t^2}{2} + ty + 4t^3\right)\, dy \\ &= \left(4t^2 z + 2t^2 \frac{z^2}{2}\right)\Bigg|_0^{4t} + 2\left(\frac{5t^2}{2} z + t^2 \frac{z^2}{2}\right)\Bigg|_0^{4t} + 4\left(\frac{t^2}{2} y + t\frac{y^2}{2} + 4t^3 y\right)\Bigg|_0^{2t} \\ &= (16t^3 + 16t^4) + 2(10t^3 + 8t^4) + 4(t^3 + 2t^3 + 8t^4) \\ &= 16t^3 + 16t^4 + 20t^3 + 16t^4 + 4t^3 + 8t^3 + 32t^4 \\ &= 48t^3 + 64t^4\end{aligned}$$

Therefore,

$$\begin{aligned}\iiint_V \frac{\partial s}{\partial t}\, dV + \oiint_S s(\mathbf{v}_S \cdot \hat{\mathbf{n}})\, dS &= 16t^4 + 48t^3 + 64t^4 \\ &= 48t^3 + 80t^4 \\ &= 16t^3(3 + 5t).\end{aligned}$$

We conclude that the Leibniz formula is verified.