Exercise 1

Show that

$$\int_0^{2\pi}\int_0^{\pi} \boldsymbol{\delta}_r \sin\theta\, d\theta\, d\phi = 0$$
$$\int_0^{2\pi}\int_0^{\pi} \boldsymbol{\delta}_r\boldsymbol{\delta}_r \sin\theta\, d\theta\, d\phi = \frac{4}{3}\pi\boldsymbol{\delta}$$

where $\boldsymbol{\delta}_r$ is the unit vector in the r direction in spherical coordinates.

Solution

To evaluate the integrals, write $\boldsymbol{\delta}_r$ in terms of the unit vectors in Cartesian coordinates.

The First Integral

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi} \boldsymbol{\delta}_r \sin\theta\, d\theta\, d\phi &= \int_0^{2\pi}\int_0^{\pi} [(\sin\theta\cos\phi)\boldsymbol{\delta}_x + (\sin\theta\sin\phi)\boldsymbol{\delta}_y + \cos\theta\boldsymbol{\delta}_z]\sin\theta\, d\theta\, d\phi \\
&= \boldsymbol{\delta}_x \int_0^{2\pi}\int_0^{\pi} \sin^2\theta\cos\phi\, d\theta\, d\phi + \boldsymbol{\delta}_y \int_0^{2\pi}\int_0^{\pi} \sin^2\theta\sin\phi\, d\theta\, d\phi \\
&\quad + \boldsymbol{\delta}_z \int_0^{2\pi}\int_0^{\pi} \sin\theta\cos\theta\, d\theta\, d\phi \\
&= \boldsymbol{\delta}_x \left(\int_0^{\pi} \sin^2\theta\, d\theta\right)\left(\int_0^{2\pi}\cos\phi\, d\phi\right) + \boldsymbol{\delta}_y \left(\int_0^{\pi} \sin^2\theta\, d\theta\right)\left(\int_0^{2\pi}\sin\phi\, d\phi\right) \\
&\quad + \boldsymbol{\delta}_z \left(\int_0^{\pi} \sin\theta\cos\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right)
\end{aligned}$$

Let $u = \sin\theta$ in the third integral in $d\theta$. Then $du = \cos\theta\, d\theta$.

$$\begin{aligned}
&= \boldsymbol{\delta}_x \left(\int_0^{\pi} \sin^2\theta\, d\theta\right)\sin\phi\Big|_0^{2\pi} + \boldsymbol{\delta}_y \left(\int_0^{\pi} \sin^2\theta\, d\theta\right)(-\cos\phi)\Big|_0^{2\pi} \\
&\quad + \boldsymbol{\delta}_z \left(\int_0^{0} u\, du\right)\left(\int_0^{2\pi} d\phi\right) \\
&= 0\boldsymbol{\delta}_x + 0\boldsymbol{\delta}_y + 0\boldsymbol{\delta}_z \\
&= \mathbf{0}
\end{aligned}$$

The Second Integral

Multiplying $\boldsymbol{\delta}_r$ and $\boldsymbol{\delta}_r$ together and distributing, we get

$$\begin{aligned}
\boldsymbol{\delta}_r\boldsymbol{\delta}_r &= [(\sin\theta\cos\phi)\boldsymbol{\delta}_x + (\sin\theta\sin\phi)\boldsymbol{\delta}_y + \cos\theta\boldsymbol{\delta}_z][(\sin\theta\cos\phi)\boldsymbol{\delta}_x + (\sin\theta\sin\phi)\boldsymbol{\delta}_y + \cos\theta\boldsymbol{\delta}_z] \\
&= \boldsymbol{\delta}_x\boldsymbol{\delta}_x \sin^2\theta\cos^2\phi + \boldsymbol{\delta}_x\boldsymbol{\delta}_y \sin^2\theta\sin\phi\cos\phi + \boldsymbol{\delta}_x\boldsymbol{\delta}_z \sin\theta\cos\theta\cos\phi \\
&\quad + \boldsymbol{\delta}_y\boldsymbol{\delta}_x \sin^2\theta\sin\phi\cos\phi + \boldsymbol{\delta}_y\boldsymbol{\delta}_y \sin^2\theta\sin^2\phi + \boldsymbol{\delta}_y\boldsymbol{\delta}_z \sin\theta\cos\theta\sin\phi \\
&\quad + \boldsymbol{\delta}_z\boldsymbol{\delta}_x \sin\theta\cos\theta\cos\phi + \boldsymbol{\delta}_z\boldsymbol{\delta}_y \sin\theta\cos\theta\sin\phi + \boldsymbol{\delta}_z\boldsymbol{\delta}_z \cos^2\theta.
\end{aligned}$$

Plugging this expression into the double integral, we get

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\boldsymbol{\delta}_r\boldsymbol{\delta}_r\sin\theta\,d\theta\,d\phi &= \int_0^{2\pi}\int_0^{\pi}[\boldsymbol{\delta}_x\boldsymbol{\delta}_x\sin^2\theta\cos^2\phi + \boldsymbol{\delta}_x\boldsymbol{\delta}_y\sin^2\theta\sin\phi\cos\phi + \boldsymbol{\delta}_x\boldsymbol{\delta}_z\sin\theta\cos\theta\cos\phi \\ &\quad + \boldsymbol{\delta}_y\boldsymbol{\delta}_x\sin^2\theta\sin\phi\cos\phi + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\sin^2\theta\sin^2\phi + \boldsymbol{\delta}_y\boldsymbol{\delta}_z\sin\theta\cos\theta\sin\phi \\ &\quad + \boldsymbol{\delta}_z\boldsymbol{\delta}_x\sin\theta\cos\theta\cos\phi + \boldsymbol{\delta}_z\boldsymbol{\delta}_y\sin\theta\cos\theta\sin\phi + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\cos^2\theta]\sin\theta\,d\theta\,d\phi \\ &= \int_0^{2\pi}\int_0^{\pi}[\boldsymbol{\delta}_x\boldsymbol{\delta}_x\sin^3\theta\cos^2\phi + \boldsymbol{\delta}_x\boldsymbol{\delta}_y\sin^3\theta\sin\phi\cos\phi + \boldsymbol{\delta}_x\boldsymbol{\delta}_z\sin^2\theta\cos\theta\cos\phi \\ &\quad + \boldsymbol{\delta}_y\boldsymbol{\delta}_x\sin^3\theta\sin\phi\cos\phi + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\sin^3\theta\sin^2\phi + \boldsymbol{\delta}_y\boldsymbol{\delta}_z\sin^2\theta\cos\theta\sin\phi \\ &\quad + \boldsymbol{\delta}_z\boldsymbol{\delta}_x\sin^2\theta\cos\theta\cos\phi + \boldsymbol{\delta}_z\boldsymbol{\delta}_y\sin^2\theta\cos\theta\sin\phi + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\sin\theta\cos^2\theta]\,d\theta\,d\phi.\end{aligned}$$

Six of the integrals evaluate to zero. Only the diagonal terms remain.

$$\begin{aligned}&= \boldsymbol{\delta}_x\boldsymbol{\delta}_x\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\cos^2\phi\,d\theta\,d\phi + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\sin^2\phi\,d\theta\,d\phi \\ &\qquad + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\int_0^{2\pi}\int_0^{\pi}\sin\theta\cos^2\theta\,d\theta\,d\phi \\ &= \boldsymbol{\delta}_x\boldsymbol{\delta}_x\left(\int_0^{\pi}\sin^3\theta\,d\theta\right)\left(\int_0^{2\pi}\cos^2\phi\,d\phi\right) + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\left(\int_0^{\pi}\sin^3\theta\,d\theta\right)\left(\int_0^{2\pi}\sin^2\phi\,d\phi\right) \\ &\qquad + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\left(\int_0^{\pi}\sin\theta\cos^2\theta\,d\theta\right)\left(\int_0^{2\pi}d\phi\right) \\ &= \boldsymbol{\delta}_x\boldsymbol{\delta}_x\left(\frac{4}{3}\right)(\pi) + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\left(\frac{4}{3}\right)(\pi) + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\left(\frac{2}{3}\right)(2\pi) \\ &= \frac{4\pi}{3}\boldsymbol{\delta}_x\boldsymbol{\delta}_x + \frac{4\pi}{3}\boldsymbol{\delta}_y\boldsymbol{\delta}_y + \frac{4\pi}{3}\boldsymbol{\delta}_z\boldsymbol{\delta}_z \\ &= \frac{4\pi}{3}\boldsymbol{\delta}\end{aligned}$$

Therefore,

$$\int_0^{2\pi}\int_0^{\pi}\boldsymbol{\delta}_r\sin\theta\,d\theta\,d\phi = 0$$

$$\int_0^{2\pi}\int_0^{\pi}\boldsymbol{\delta}_r\boldsymbol{\delta}_r\sin\theta\,d\theta\,d\phi = \frac{4}{3}\pi\boldsymbol{\delta}.$$