Exercise 2

Verify that in spherical coordinates $\boldsymbol{\delta} = \boldsymbol{\delta}_r\boldsymbol{\delta}_r + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta + \boldsymbol{\delta}_\phi\boldsymbol{\delta}_\phi$.

Solution

The Left-hand Side

In Cartesian coordinates we have for the unit tensor

$$\begin{aligned}\boldsymbol{\delta} &= \sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_i\boldsymbol{\delta}_j\delta_{ij} \\ &= \sum_{i=1}^{3}\boldsymbol{\delta}_i\boldsymbol{\delta}_i \\ &= \boldsymbol{\delta}_1\boldsymbol{\delta}_1 + \boldsymbol{\delta}_2\boldsymbol{\delta}_2 + \boldsymbol{\delta}_3\boldsymbol{\delta}_3\end{aligned}$$

or

$$= \boldsymbol{\delta}_x\boldsymbol{\delta}_x + \boldsymbol{\delta}_y\boldsymbol{\delta}_y + \boldsymbol{\delta}_z\boldsymbol{\delta}_z.$$

The Right-hand Side

In Cartesian coordinates the unit vectors in spherical coordinates are

$$\begin{aligned}\boldsymbol{\delta}_r &= \sin\theta\cos\phi\boldsymbol{\delta}_x + \sin\theta\sin\phi\boldsymbol{\delta}_y + \cos\theta\boldsymbol{\delta}_z \\ \boldsymbol{\delta}_\theta &= \cos\theta\cos\phi\boldsymbol{\delta}_x + \cos\theta\sin\phi\boldsymbol{\delta}_y - \sin\theta\boldsymbol{\delta}_z \\ \boldsymbol{\delta}_\phi &= -\sin\phi\boldsymbol{\delta}_x + \cos\phi\boldsymbol{\delta}_y + 0\boldsymbol{\delta}_z,\end{aligned}$$

so the unit dyads $\boldsymbol{\delta}_r\boldsymbol{\delta}_r$, $\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta$, and $\boldsymbol{\delta}_\phi\boldsymbol{\delta}_\phi$ are

$$\begin{aligned}\boldsymbol{\delta}_r\boldsymbol{\delta}_r &= [(\sin\theta\cos\phi)\boldsymbol{\delta}_x + (\sin\theta\sin\phi)\boldsymbol{\delta}_y + \cos\theta\boldsymbol{\delta}_z][(\sin\theta\cos\phi)\boldsymbol{\delta}_x + (\sin\theta\sin\phi)\boldsymbol{\delta}_y + \cos\theta\boldsymbol{\delta}_z] \\ &= \boldsymbol{\delta}_x\boldsymbol{\delta}_x\sin^2\theta\cos^2\phi + \boldsymbol{\delta}_x\boldsymbol{\delta}_y\sin^2\theta\sin\phi\cos\phi + \boldsymbol{\delta}_x\boldsymbol{\delta}_z\sin\theta\cos\theta\cos\phi \\ &\quad + \boldsymbol{\delta}_y\boldsymbol{\delta}_x\sin^2\theta\sin\phi\cos\phi + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\sin^2\theta\sin^2\phi + \boldsymbol{\delta}_y\boldsymbol{\delta}_z\sin\theta\cos\theta\sin\phi \\ &\quad + \boldsymbol{\delta}_z\boldsymbol{\delta}_x\sin\theta\cos\theta\cos\phi + \boldsymbol{\delta}_z\boldsymbol{\delta}_y\sin\theta\cos\theta\sin\phi + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\cos^2\theta\end{aligned}$$

$$\begin{aligned}\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta &= [(\cos\theta\cos\phi)\boldsymbol{\delta}_x + (\cos\theta\sin\phi)\boldsymbol{\delta}_y - \sin\theta\boldsymbol{\delta}_z][(\cos\theta\cos\phi)\boldsymbol{\delta}_x + (\cos\theta\sin\phi)\boldsymbol{\delta}_y - \sin\theta\boldsymbol{\delta}_z] \\ &= \boldsymbol{\delta}_x\boldsymbol{\delta}_x\cos^2\theta\cos^2\phi + \boldsymbol{\delta}_x\boldsymbol{\delta}_y\cos^2\theta\sin\phi\cos\phi - \boldsymbol{\delta}_x\boldsymbol{\delta}_z\sin\theta\cos\theta\cos\phi \\ &\quad + \boldsymbol{\delta}_y\boldsymbol{\delta}_x\cos^2\theta\sin\phi\cos\phi + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\cos^2\theta\sin^2\phi - \boldsymbol{\delta}_y\boldsymbol{\delta}_z\sin\theta\cos\theta\sin\phi \\ &\quad - \boldsymbol{\delta}_z\boldsymbol{\delta}_x\sin\theta\cos\theta\cos\phi - \boldsymbol{\delta}_z\boldsymbol{\delta}_y\sin\theta\cos\theta\sin\phi + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\sin^2\theta\end{aligned}$$

$$\begin{aligned}\boldsymbol{\delta}_\phi\boldsymbol{\delta}_\phi &= [-\sin\phi\boldsymbol{\delta}_x + \cos\phi\boldsymbol{\delta}_y + 0\boldsymbol{\delta}_z][-\sin\phi\boldsymbol{\delta}_x + \cos\phi\boldsymbol{\delta}_y + 0\boldsymbol{\delta}_z] \\ &= \boldsymbol{\delta}_x\boldsymbol{\delta}_x\sin^2\phi - \boldsymbol{\delta}_x\boldsymbol{\delta}_y\sin\phi\cos\phi - \boldsymbol{\delta}_y\boldsymbol{\delta}_x\sin\phi\cos\phi + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\cos^2\phi.\end{aligned}$$

Adding them all together, we find that $\boldsymbol{\delta}_r\boldsymbol{\delta}_r + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta + \boldsymbol{\delta}_\phi\boldsymbol{\delta}_\phi = \boldsymbol{\delta}_x\boldsymbol{\delta}_x + \boldsymbol{\delta}_y\boldsymbol{\delta}_y + \boldsymbol{\delta}_z\boldsymbol{\delta}_z$. Therefore, $\boldsymbol{\delta} = \boldsymbol{\delta}_r\boldsymbol{\delta}_r + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta + \boldsymbol{\delta}_\phi\boldsymbol{\delta}_\phi$.