Exercise 1

If $\mathbf{r}$ is the instantaneous position vector for a particle, show that the velocity and acceleration of the particle are given by (use Eq. A.7-2):

$$\mathbf{v} = \frac{d}{dt}\mathbf{r} = \boldsymbol{\delta}_r \dot{r} + \boldsymbol{\delta}_\theta r\dot{\theta} + \boldsymbol{\delta}_z \dot{z} \tag{A.7-34}$$

$$\mathbf{a} = \boldsymbol{\delta}_r(\ddot{r} - r\dot{\theta}^2) + \boldsymbol{\delta}_\theta(r\ddot{\theta} + 2\dot{r}\dot{\theta}) + \boldsymbol{\delta}_z \ddot{z} \tag{A.7-35}$$

in cylindrical coordinates. The dots indicate time derivatives of the coordinates.

Solution

If $\mathbf{r}$ represents the position vector, then we have

$$\mathbf{r} = x\boldsymbol{\delta}_x + y\boldsymbol{\delta}_y + z\boldsymbol{\delta}_z$$

to start with. Change to polar coordinates by making the substitutions, $x = r\cos\theta$ and $y = r\sin\theta$ and $z = z$. As a result, the unit vectors become

$$\begin{aligned}
\boldsymbol{\delta}_x &= \cos\theta\boldsymbol{\delta}_r - \sin\theta\boldsymbol{\delta}_\theta \\
\boldsymbol{\delta}_y &= \sin\theta\boldsymbol{\delta}_r + \cos\theta\boldsymbol{\delta}_\theta \\
\boldsymbol{\delta}_z &= \boldsymbol{\delta}_z,
\end{aligned}$$

and we get

$$\begin{aligned}
\mathbf{r} &= r\cos\theta(\cos\theta\boldsymbol{\delta}_r - \sin\theta\boldsymbol{\delta}_\theta) + r\sin\theta(\sin\theta\boldsymbol{\delta}_r + \cos\theta\boldsymbol{\delta}_\theta) + z\boldsymbol{\delta}_z \\
&= \boldsymbol{\delta}_r(r\cos^2\theta + r\sin^2\theta) + \boldsymbol{\delta}_\theta(-\cancel{r\sin\theta\cos\theta} + \cancel{r\sin\theta\cos\theta}) + z\boldsymbol{\delta}_z \\
&= r\boldsymbol{\delta}_r + z\boldsymbol{\delta}_z.
\end{aligned}$$

Now that we have the position vector in polar coordinates, we're ready to take the time derivative of it.

$$\begin{aligned}
\frac{d\mathbf{r}}{dt} &= \frac{d}{dt}(r\boldsymbol{\delta}_r + z\boldsymbol{\delta}_z) \\
&= \frac{d}{dt}(r\boldsymbol{\delta}_r) + \frac{d}{dt}(z\boldsymbol{\delta}_z) \\
&= \frac{dr}{dt}\boldsymbol{\delta}_r + r\frac{d\boldsymbol{\delta}_r}{dt} + \frac{dz}{dt}\boldsymbol{\delta}_z + z\frac{d\boldsymbol{\delta}_z}{dt}
\end{aligned}$$

Eq. A.7-2 gives the partial derivatives with respect to θ of the unit vectors in cylindrical coordinates,

$$\frac{\partial\boldsymbol{\delta}_r}{\partial\theta} = \boldsymbol{\delta}_\theta \qquad \frac{\partial\boldsymbol{\delta}_\theta}{\partial\theta} = -\boldsymbol{\delta}_r \qquad \frac{\partial\boldsymbol{\delta}_z}{\partial\theta} = \mathbf{0}. \tag{A.7-2}$$

We can use the chain rule to make use of these.

$$\begin{aligned}
\frac{d\mathbf{r}}{dt} &= \frac{dr}{dt}\boldsymbol{\delta}_r + r\frac{d\theta}{dt}\frac{\partial\boldsymbol{\delta}_r}{\partial\theta} + \frac{dz}{dt}\boldsymbol{\delta}_z + z\frac{d\theta}{dt}\frac{\partial\boldsymbol{\delta}_z}{\partial\theta} \\
&= \frac{dr}{dt}\boldsymbol{\delta}_r + r\frac{d\theta}{dt}(\boldsymbol{\delta}_\theta) + \frac{dz}{dt}\boldsymbol{\delta}_z + z\frac{d\theta}{dt}(\mathbf{0})
\end{aligned}$$

Therefore,

$$\mathbf{v} = \frac{d}{dt}\mathbf{r} = \boldsymbol{\delta}_r\dot{r} + \boldsymbol{\delta}_\theta r\dot{\theta} + \boldsymbol{\delta}_z\dot{z}.$$

Take a second derivative to obtain the acceleration vector.

$$\begin{aligned}
\mathbf{a} &= \frac{d}{dt}\mathbf{v} \\
&= \frac{d}{dt}\left(\frac{dr}{dt}\boldsymbol{\delta}_r + r\frac{d\theta}{dt}\boldsymbol{\delta}_\theta + \frac{dz}{dt}\boldsymbol{\delta}_z\right) \\
&= \frac{d}{dt}\left(\frac{dr}{dt}\boldsymbol{\delta}_r\right) + \frac{d}{dt}\left(r\frac{d\theta}{dt}\boldsymbol{\delta}_\theta\right) + \frac{d}{dt}\left(\frac{dz}{dt}\boldsymbol{\delta}_z\right) \\
&= \frac{d^2r}{dt^2}\boldsymbol{\delta}_r + \frac{dr}{dt}\frac{d\boldsymbol{\delta}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\boldsymbol{\delta}_\theta + r\frac{d^2\theta}{dt^2}\boldsymbol{\delta}_\theta + r\frac{d\theta}{dt}\frac{d\boldsymbol{\delta}_\theta}{dt} + \frac{d^2z}{dt^2}\boldsymbol{\delta}_z + \frac{dz}{dt}\frac{d\boldsymbol{\delta}_z}{dt}
\end{aligned}$$

Use the chain rule in order to use the relations in Eq. A.7-2.

$$= \frac{d^2r}{dt^2}\boldsymbol{\delta}_r + \frac{dr}{dt}\frac{d\theta}{dt}\frac{\partial\boldsymbol{\delta}_r}{\partial\theta} + \frac{dr}{dt}\frac{d\theta}{dt}\boldsymbol{\delta}_\theta + r\frac{d^2\theta}{dt^2}\boldsymbol{\delta}_\theta + r\frac{d\theta}{dt}\frac{d\theta}{dt}\frac{\partial\boldsymbol{\delta}_\theta}{\partial\theta} + \frac{d^2z}{dt^2}\boldsymbol{\delta}_z + \frac{dz}{dt}\frac{d\theta}{dt}\frac{\partial\boldsymbol{\delta}_z}{\partial\theta}$$

Now we can use them.

$$= \frac{d^2r}{dt^2}\boldsymbol{\delta}_r + \frac{dr}{dt}\frac{d\theta}{dt}(\boldsymbol{\delta}_\theta) + \frac{dr}{dt}\frac{d\theta}{dt}\boldsymbol{\delta}_\theta + r\frac{d^2\theta}{dt^2}\boldsymbol{\delta}_\theta + r\frac{d\theta}{dt}\frac{d\theta}{dt}(-\boldsymbol{\delta}_r) + \frac{d^2z}{dt^2}\boldsymbol{\delta}_z + \frac{dz}{dt}\frac{d\theta}{dt}(\mathbf{0})$$

Factor the unit vectors.

$$= \boldsymbol{\delta}_r\left(\frac{d^2r}{dt^2} - r\frac{d\theta}{dt}\frac{d\theta}{dt}\right) + \boldsymbol{\delta}_\theta\left(\frac{dr}{dt}\frac{d\theta}{dt} + \frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right) + \frac{d^2z}{dt^2}\boldsymbol{\delta}_z$$

Therefore,

$$\mathbf{a} = \boldsymbol{\delta}_r(\ddot{r} - r\dot{\theta}^2) + \boldsymbol{\delta}_\theta(r\ddot{\theta} + 2\dot{r}\dot{\theta}) + \boldsymbol{\delta}_z\ddot{z}.$$