Problem 1B.1

Velocity profiles and the stress components $\boldsymbol{\tau}_{ij}$. For each of the following velocity distributions, draw a meaningful sketch showing the flow pattern. Then find all the components of $\boldsymbol{\tau}$ and $\rho\mathbf{vv}$ for the Newtonian fluid. The parameter b is a constant.

(a) $v_x = by$, $v_y = 0$, $v_z = 0$

(b) $v_x = by$, $v_y = bx$, $v_z = 0$

(c) $v_x = -by$, $v_y = bx$, $v_z = 0$

(d) $v_x = -\frac{1}{2}bx$, $v_y = -\frac{1}{2}by$, $v_z = bz$

Solution

The second-order tensors, $\boldsymbol{\tau}$ and $\rho\mathbf{vv}$, are

$$\boldsymbol{\tau} = \sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j\tau_{ij}$$

$$\rho\mathbf{vv} = \rho\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i v_i\right)\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j v_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_i\boldsymbol{\delta}_j\rho v_i v_j.$$

Write out the terms of the sums.

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{\delta}_1\boldsymbol{\delta}_1\tau_{11} + \boldsymbol{\delta}_1\boldsymbol{\delta}_2\tau_{12} + \boldsymbol{\delta}_1\boldsymbol{\delta}_3\tau_{13} \\ &\quad + \boldsymbol{\delta}_2\boldsymbol{\delta}_1\tau_{21} + \boldsymbol{\delta}_2\boldsymbol{\delta}_2\tau_{22} + \boldsymbol{\delta}_2\boldsymbol{\delta}_3\tau_{23} \\ &\quad + \boldsymbol{\delta}_3\boldsymbol{\delta}_1\tau_{31} + \boldsymbol{\delta}_3\boldsymbol{\delta}_2\tau_{32} + \boldsymbol{\delta}_3\boldsymbol{\delta}_3\tau_{33} \\ \rho\mathbf{vv} &= \boldsymbol{\delta}_1\boldsymbol{\delta}_1\rho v_1 v_1 + \boldsymbol{\delta}_1\boldsymbol{\delta}_2\rho v_1 v_2 + \boldsymbol{\delta}_1\boldsymbol{\delta}_3\rho v_1 v_3 \\ &\quad + \boldsymbol{\delta}_2\boldsymbol{\delta}_1\rho v_2 v_1 + \boldsymbol{\delta}_2\boldsymbol{\delta}_2\rho v_2 v_2 + \boldsymbol{\delta}_2\boldsymbol{\delta}_3\rho v_2 v_3 \\ &\quad + \boldsymbol{\delta}_3\boldsymbol{\delta}_1\rho v_3 v_1 + \boldsymbol{\delta}_3\boldsymbol{\delta}_2\rho v_3 v_2 + \boldsymbol{\delta}_3\boldsymbol{\delta}_3\rho v_3 v_3\end{aligned}$$

$\boldsymbol{\tau}$ and $\rho\mathbf{vv}$ each have nine components. The fact that we have a Newtonian fluid means the viscous stresses are of the form,

$$\tau_{ij} = -\mu\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right) + \left(\frac{2}{3}\mu - \kappa\right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\delta_{ij},$$

so

$$\begin{aligned}
\tau_{11} &= -2\mu\frac{\partial v_x}{\partial x} + \left(\frac{2}{3}\mu - \kappa\right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \\
\tau_{22} &= -2\mu\frac{\partial v_y}{\partial y} + \left(\frac{2}{3}\mu - \kappa\right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \\
\tau_{33} &= -2\mu\frac{\partial v_z}{\partial z} + \left(\frac{2}{3}\mu - \kappa\right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \\
\tau_{12} &= \tau_{21} = -\mu\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) \\
\tau_{13} &= \tau_{31} = -\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) \\
\tau_{23} &= \tau_{32} = -\mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right).
\end{aligned}$$

Part (a)

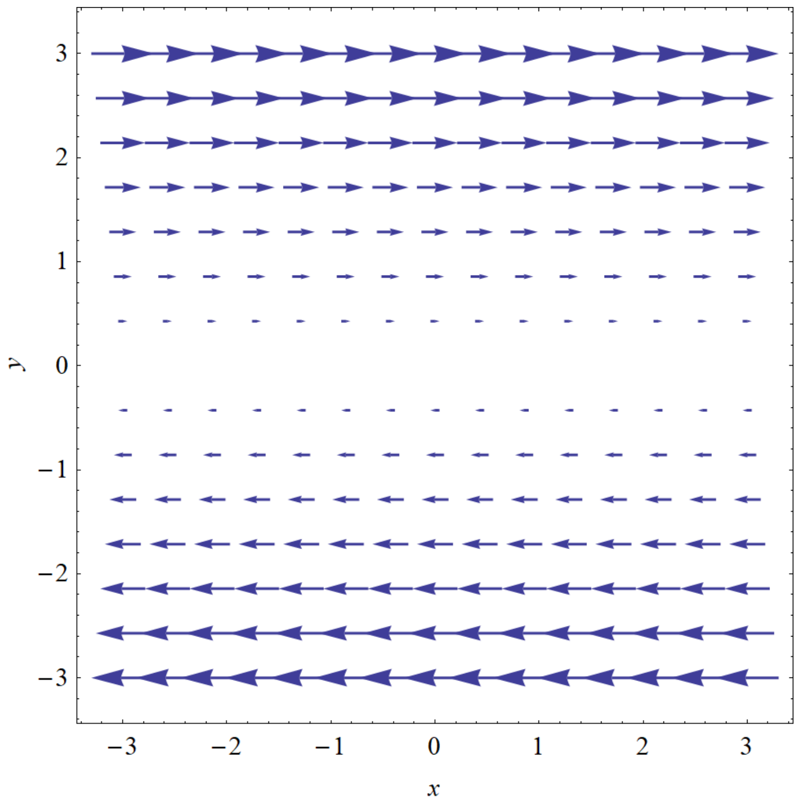

Figure 1: Plot of the velocity field with $b = 1$ for $-3 < x < 3$ and $-3 < y < 3$.

$$\begin{aligned}
\boldsymbol{\tau} &= -\mu b \boldsymbol{\delta}_x \boldsymbol{\delta}_y - \mu b \boldsymbol{\delta}_y \boldsymbol{\delta}_x \\
\rho \mathbf{v}\mathbf{v} &= \rho b^2 y^2 \boldsymbol{\delta}_x \boldsymbol{\delta}_x
\end{aligned}$$

Part (b)

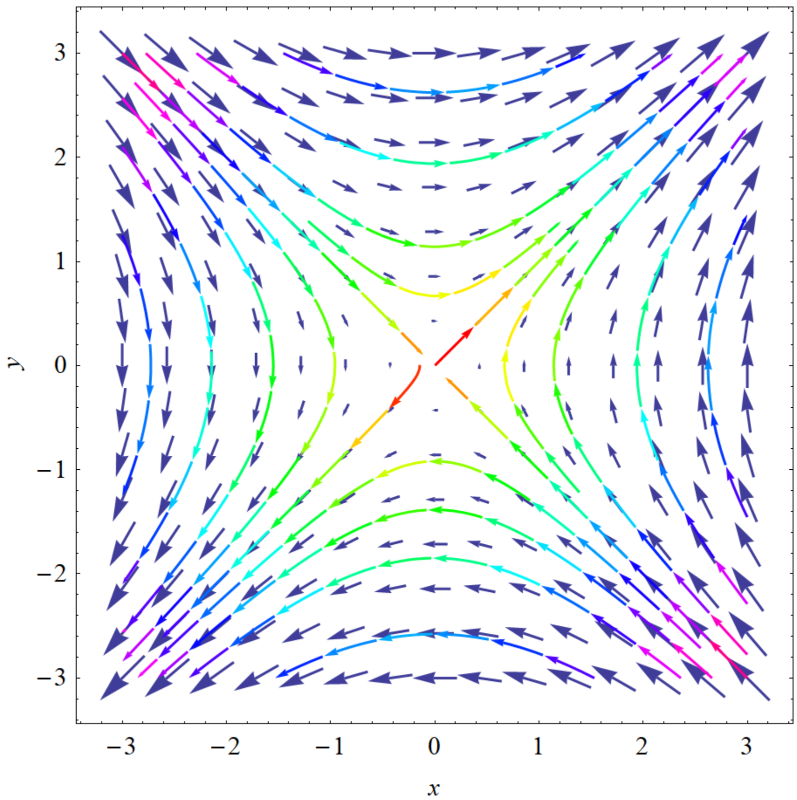

Figure 2: Plot of the velocity field with streamlines and $b = 1$ for $-3 < x < 3$ and $-3 < y < 3$.

$$\boldsymbol{\tau} = -2\mu b \boldsymbol{\delta}_x \boldsymbol{\delta}_y - 2\mu b \boldsymbol{\delta}_y \boldsymbol{\delta}_x$$
$$\rho \mathbf{v}\mathbf{v} = \rho b^2 y^2 \boldsymbol{\delta}_x \boldsymbol{\delta}_x + \rho b^2 xy \boldsymbol{\delta}_x \boldsymbol{\delta}_y + \rho b^2 xy \boldsymbol{\delta}_y \boldsymbol{\delta}_x + \rho b^2 x^2 \boldsymbol{\delta}_y \boldsymbol{\delta}_y$$

Part (c)

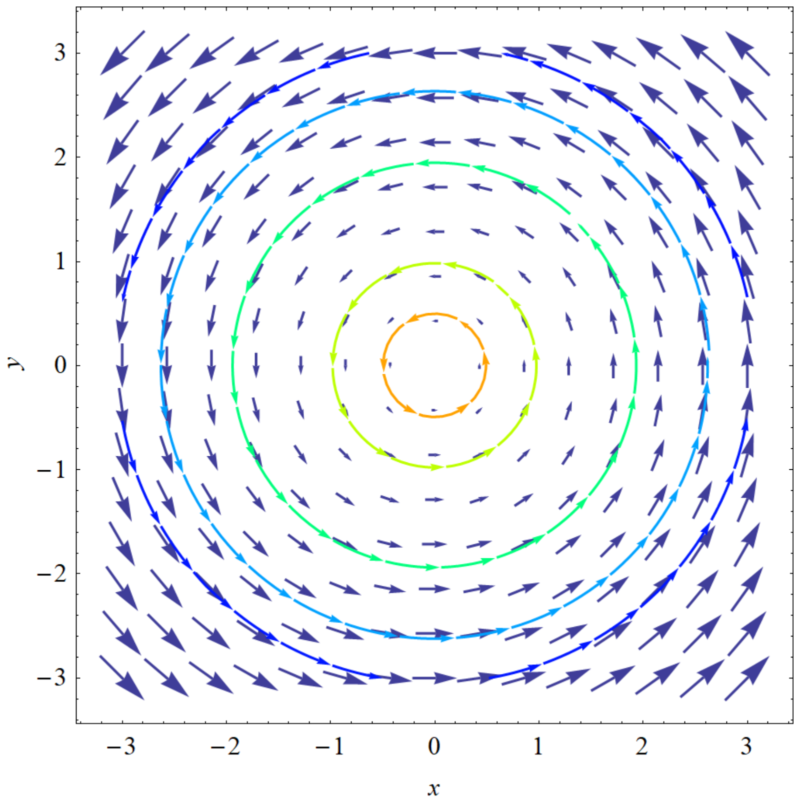

Figure 3: Plot of the velocity field with streamlines and $b = 1$ for $-3 < x < 3$ and $-3 < y < 3$.

$$\boldsymbol{\tau} = \mathbf{0}$$

$$\rho\mathbf{vv} = \rho b^2 y^2 \boldsymbol{\delta}_x \boldsymbol{\delta}_x - \rho b^2 xy \boldsymbol{\delta}_x \boldsymbol{\delta}_y - \rho b^2 xy \boldsymbol{\delta}_y \boldsymbol{\delta}_x + \rho b^2 x^2 \boldsymbol{\delta}_y \boldsymbol{\delta}_y$$

Part (d)

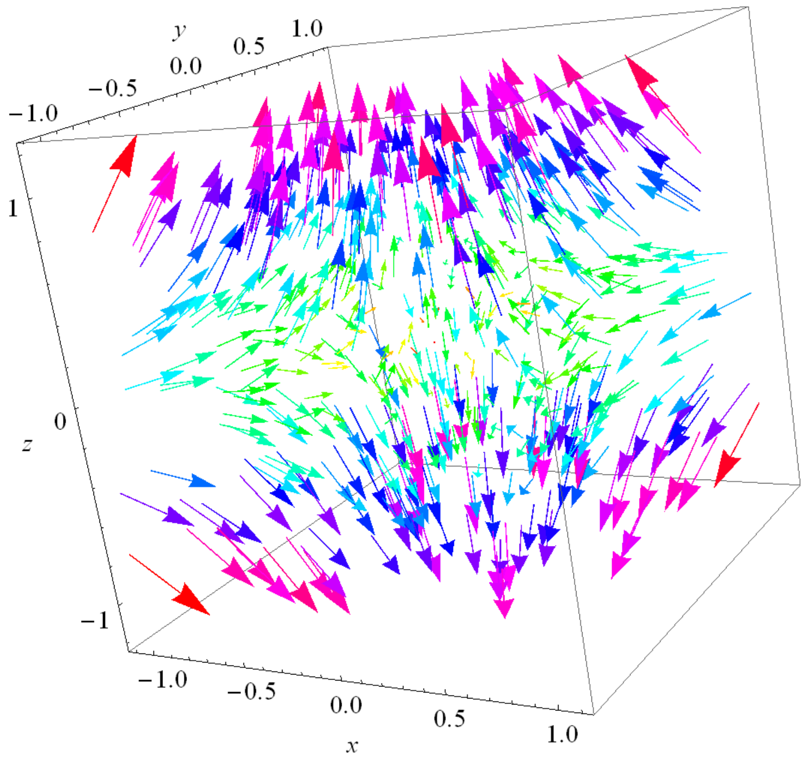

Figure 4: Plot of the velocity field with $b = 1$ for $-1 < x < 1$ and $-1 < y < 1$ and $-1 < z < 1$.

$$\begin{aligned}
\boldsymbol{\tau} &= \mu b \boldsymbol{\delta}_x \boldsymbol{\delta}_x + \mu b \boldsymbol{\delta}_y \boldsymbol{\delta}_y - 2\mu b \boldsymbol{\delta}_z \boldsymbol{\delta}_z \\
\rho \mathbf{v}\mathbf{v} &= \frac{\rho}{4} b^2 x^2 \boldsymbol{\delta}_x \boldsymbol{\delta}_x + \frac{\rho}{4} b^2 xy \boldsymbol{\delta}_x \boldsymbol{\delta}_y - \frac{\rho}{2} b^2 xz \boldsymbol{\delta}_x \boldsymbol{\delta}_z \\
&\quad + \frac{\rho}{4} b^2 xy \boldsymbol{\delta}_y \boldsymbol{\delta}_x + \frac{\rho}{4} b^2 y^2 \boldsymbol{\delta}_y \boldsymbol{\delta}_y - \frac{\rho}{2} b^2 yz \boldsymbol{\delta}_y \boldsymbol{\delta}_z \\
&\quad - \frac{\rho}{2} b^2 xz \boldsymbol{\delta}_z \boldsymbol{\delta}_x - \frac{\rho}{2} b^2 yz \boldsymbol{\delta}_z \boldsymbol{\delta}_y + \rho b^2 z^2 \boldsymbol{\delta}_z \boldsymbol{\delta}_z
\end{aligned}$$