Problem 1B.2

A fluid in a state of rigid rotation.

(a) Verify that the velocity distribution (c) in Problem 1B.1 describes a fluid in a state of pure rotation; that is, the fluid is rotating like a rigid body. What is the angular velocity of rotation?

(b) For that flow pattern evaluate the symmetric and antisymmetric combinations of velocity derivatives:

(i) $(\partial v_y/\partial x) + (\partial v_x/\partial y)$

(ii) $(\partial v_y/\partial x) - (\partial v_x/\partial y)$

(c) Discuss the results of (b) in connection with the development in §1.2.

Solution

Part (a)

The velocity distribution in Problem 1B.1 (c) is $v_x = -by$, $v_y = bx$, and $v_z = 0$.

$$\mathbf{v} = -by\boldsymbol{\delta}_x + bx\boldsymbol{\delta}_y + 0\boldsymbol{\delta}_z$$

To verify that the fluid is purely rotating, switch to polar coordinates. Use the variable transformations,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \end{aligned}$$

and use the unit vector transformations (from Appendix A.6 on page 827),

$$\begin{aligned} \boldsymbol{\delta}_x &= \cos\theta\boldsymbol{\delta}_r - \sin\theta\boldsymbol{\delta}_\theta + 0\boldsymbol{\delta}_z \\ \boldsymbol{\delta}_y &= \sin\theta\boldsymbol{\delta}_r + \cos\theta\boldsymbol{\delta}_\theta + 0\boldsymbol{\delta}_z \\ \boldsymbol{\delta}_z &= 0\boldsymbol{\delta}_r + 0\boldsymbol{\delta}_\theta + 1\boldsymbol{\delta}_z. \end{aligned}$$

Making these substitutions, the velocity becomes

$$\mathbf{v} = -br\sin\theta(\cos\theta\boldsymbol{\delta}_r - \sin\theta\boldsymbol{\delta}_\theta) + br\cos\theta(\sin\theta\boldsymbol{\delta}_r + \cos\theta\boldsymbol{\delta}_\theta) + 0\boldsymbol{\delta}_z.$$

Expand the result and factor the polar unit vectors.

$$\mathbf{v} = (-\cancel{br\sin\theta\cos\theta} + \cancel{br\sin\theta\cos\theta})\boldsymbol{\delta}_r + (br\sin^2\theta + br\cos^2\theta)\boldsymbol{\delta}_\theta + 0\boldsymbol{\delta}_z$$

Apply $\sin^2\theta + \cos^2\theta = 1$ to get the velocity in polar coordinates.

$$\mathbf{v} = br\boldsymbol{\delta}_\theta$$

Because the velocity only has a θ-component, the fluid is in pure rotation. The magnitude of the angular velocity is obtained by dividing the θ-component of the velocity by the radius.

$$|\boldsymbol{\omega}| = \frac{v_\theta}{r} = \frac{br}{r} = b$$

Since the sign of the θ-component is positive, the fluid is travelling counterclockwise about the origin. This means the angular velocity vector points in the positive z-direction. Therefore,

$$\boldsymbol{\omega} = b\boldsymbol{\delta}_z.$$

Part (b)

$$\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} = b - b = 0$$
$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = b - (-b) = 2b$$

Part (c)

We see that a symmetric combination of the velocity gradients is zero but is not zero for an antisymmetric combination. This is why we omit the antisymmetric combinations in the generalized law of viscosity—we require $\boldsymbol{\tau} = \mathbf{0}$ for the case that the fluid is in a state of pure rotation.