Problem 1C.2

The wall collision frequency. It is desired to find the frequency $Z$ with which the molecules in an ideal gas strike a unit area of a wall from one side only. The gas is at rest and at equilibrium with a temperature $T$ and the number density of the molecules is $n$. All molecules have a mass $m$. All molecules in the region $x < 0$ with $u_x > 0$ will hit an area $S$ in the $yz$-plane in a short time $\Delta t$ if they are in the volume $Su_x \Delta t$. The number of wall collisions per unit area per unit time will be

$$Z = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} (Su_x \Delta t)f(u_x, u_y, u_z)du_x du_y du_z}{S \Delta t}$$

$$= n \left( \frac{m}{2\pi kT} \right)^{3/2} \left( \int_{0}^{+\infty} u_x \exp(-mu_x^2/2kT) du_x \right) \left( \int_{-\infty}^{+\infty} \exp(-mu_y^2/2kT) du_y \right) \left( \int_{-\infty}^{+\infty} \exp(-mu_z^2/2kT) du_z \right)$$

$$= n \sqrt{\frac{kT}{2\pi m}} = \frac{1}{4} n \bar{u}_x$$

(1C.2-1)

Verify the above development.

Solution

Suppose we have an $xyz$-coordinate system immersed in an ideal gas. In order for molecules to strike the $yz$-plane within a certain time interval $\Delta t$, the molecules have to be within a distance of $u_x \Delta t$ to do so, $u_x$ being the $x$-component of their velocity. If we consider a small area on the $yz$-plane $S$, then all the molecules within the volume $Su_x \Delta t$ will collide with this area. The Maxwell–Boltzmann distribution function,

$$f(u_x, u_y, u_z) = n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left( \frac{-mu_x^2}{2kT} \right),$$

(1)

allows us to find the number of molecules in an ideal gas at rest between certain velocities. $f(u_x, u_y, u_z) du_x du_y du_z$ represents the number of molecules per unit volume that has velocity between $\{u_x, u_y, u_z\}$ and $\{u_x + du_x, u_y + du_y, u_z + du_z\}$. So then this quantity times the volume $(Su_x \Delta t)f(u_x, u_y, u_z) du_x du_y du_z$ represents the number of molecules in the volume $Su_x \Delta t$ that has velocity between $\{u_x, u_y, u_z\}$ and $\{u_x + du_x, u_y + du_y, u_z + du_z\}$. Integrating over all possible velocities, we obtain the total number of molecules within the volume that will collide with the surface.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{+\infty} (Su_x \Delta t)f(u_x, u_y, u_z)du_x du_y du_z$$

The molecules are assumed to travel from left to right only, so the integral in $du_x$ only goes from $0$ to $\infty$. Dividing this by the area $S$, we get the number of collisions per unit area.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{+\infty} (Su_x \Delta t)f(u_x, u_y, u_z)du_x du_y du_z$$

$$S$$

Dividing this by the time interval $\Delta t$, we get the number of collisions per unit area per unit time $Z$.

$$Z = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{+\infty} (Su_x \Delta t)f(u_x, u_y, u_z)du_x du_y du_z}{S \Delta t}$$

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Bring out the constant from the integral and cancel it with the one in the denominator.

\[
Z = \frac{S \Delta t}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} u_x f(u_x, u_y, u_z) \, du_x \, du_y \, du_z}
\]

Substitute the formula for \( f(u_x, u_y, u_z) \) in equation (1).

\[
Z = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} u_x n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left( -\frac{mu_x^2}{2kT} \right) \, du_x \, du_y \, du_z.
\]

Let \( a = m/2kT \) to make the integral easier to write and substitute \( u^2 = u_x^2 + u_y^2 + u_z^2 \).

\[
Z = n \left( \frac{a}{\pi} \right)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} u_x e^{-a(u_x^2 + u_y^2 + u_z^2)} \, du_x \, du_y \, du_z.
\]

Split up the triple integral into three single integrals.

\[
Z = n \left( \frac{a}{\pi} \right)^{3/2} \left( \int_{0}^{\infty} u_x e^{-au_x^2} \, du_x \right) \left( \int_{-\infty}^{\infty} e^{-au_y^2} \, du_y \right) \left( \int_{-\infty}^{\infty} e^{-au_z^2} \, du_z \right)
\]

The integrals in \( du_y \) and \( du_z \) are known. Use the substitution,

\[
v = au_x^2
\]

\[
\frac{dv}{2a} = u_x \, du_x,
\]

to solve the first one.

\[
Z = n \left( \frac{a}{\pi} \right)^{3/2} \left( \int_{0}^{\infty} e^{-v} \, dv \right) \left( \frac{\sqrt{\pi}}{\sqrt{a}} \right) \left( \frac{\sqrt{\pi}}{\sqrt{a}} \right)
\]

Evaluate it.

\[
Z = n \left( \frac{a}{\pi} \right)^{3/2} \left( \frac{1}{2a} \right) \left( \frac{\sqrt{\pi}}{\sqrt{a}} \right) \left( \frac{\sqrt{\pi}}{\sqrt{a}} \right)
\]

Simplify the result.

\[
Z = \frac{n}{2} \frac{1}{\sqrt{\pi a}}
\]

Substitute the expression for \( a \).

\[
Z = \frac{n}{2} \sqrt{\frac{2kT}{\pi m}} = \frac{n}{4} \sqrt{\frac{8kT}{\pi m}}
\]

Therefore,

\[
Z = \frac{1}{4} \bar{u}
\]

where \( \bar{u} \) is the mean molecular speed.