Problem 1C.3

Pressure of an ideal gas. It is desired to get the pressure exerted by an ideal gas on a wall by accounting for the rate of momentum transfer from the molecules to the wall, assuming specular reflections.

(a) When a molecule traveling with a velocity \( u \) collides with a wall, its incoming velocity components are \( u_x, u_y, u_z \), and after a specular reflection at the wall, its components are \(-u_x, u_y, u_z\). Thus the net momentum transmitted to the wall by a molecule is \( 2mu_x \). The molecules that have an \( x \)-component of the velocity equal to \( u_x \), and that will collide with the wall during a small time interval \( \Delta t \), must be within the volume \( Su_x \Delta t \). How many molecules with velocity components in the range from \( u_x \), \( u_y \), \( u_z \) to \( u_x + \Delta u_x \), \( u_y + \Delta u_y \), \( u_z + \Delta u_z \) will hit an area \( S \) of the wall with a velocity \( u_x \) within a time interval \( \Delta t \)? It will be \( f(u_x, u_y, u_z)du_x du_y du_z \times Su_x \Delta t \). Then the pressure exerted on the wall by the gas will be

\[
p = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} (Su_x \Delta t)(2mu_x)f(u_x, u_y, u_z)du_x du_y du_z}{S\Delta t}
\]

(1C.3-1)

Explain carefully how this expression is constructed. Verify that this relation is dimensionally correct.

(b) Insert Eq. 1C.1-1 for the Maxwell–Boltzmann equilibrium distribution into Eq. 1C.3-1 and perform the integration. Verify that this procedure leads to \( p = n\kappa T \), the ideal gas law.

Solution

Part (a)

Suppose we have an \( xyz \)-coordinate system immersed in an ideal gas. In order for molecules to strike the \( yz \)-plane within a certain time interval \( \Delta t \), the molecules have to be within a distance of \( u_x \Delta t \) to do so, \( u_x \) being the \( x \)-component of their velocity. If we consider a small area on the \( yz \)-plane \( S \), then all the molecules within the volume \( Su_x \Delta t \) will collide with this area. The Maxwell–Boltzmann distribution function,

\[
f(u_x, u_y, u_z) = n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left( -\frac{mu^2}{2kT} \right),
\]

allows us to find the number of molecules in an ideal gas at rest between certain velocities. \( f(u_x, u_y, u_z)du_x du_y du_z \) represents the number of molecules per unit volume that has velocity between \( \langle u_x, u_y, u_z \rangle \) and \( \langle u_x + du_x, u_y + du_y, u_z + du_z \rangle \). So then this quantity times the volume \( (Su_x \Delta t)f(u_x, u_y, u_z)du_x du_y du_z \) represents the number of molecules in the volume \( Su_x \Delta t \) that has velocity between \( \langle u_x, u_y, u_z \rangle \) and \( \langle u_x + du_x, u_y + du_y, u_z + du_z \rangle \). The pressure on the surface is the force divided by area.

\[
p = \frac{F_x}{S}
\]

From Newton’s second law we know that \( F_x = ma_x = dP_x/dt \), where \( P_x \) is the component of momentum in the \( x \)-direction.

\[
p = \frac{1}{S} \frac{dP_x}{dt}
\]


www.stemjock.com
Since we are considering a finite time interval of $\Delta t$, we have

$$p = \frac{1}{S} \frac{\Delta P_x}{\Delta t}.$$ 

The total change in momentum is obtained by multiplying the total number of molecules by the change in momentum of each molecule.

$$p = \frac{1}{S \Delta t} \text{(total # of molecules} \times \text{change in momentum of each molecule}).$$

We obtain the total number of molecules in the volume by integrating

$$(S u_x \Delta t) f(u_x, u_y, u_z) \, du_x \, du_y \, du_z$$

over all possible velocities. As explained in the problem statement, the change in momentum of one molecule is $2m u_x$. Thus,

$$p = \frac{1}{S \Delta t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} (2m u_x) \, (S u_x \Delta t) f(u_x, u_y, u_z) \, du_x \, du_y \, du_z.$$ 

The molecules are assumed to travel from left to right only, so the integral in $du_x$ only goes from 0 to $\infty$. Therefore,

$$p = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} (S u_x \Delta t) (2m u_x) f(u_x, u_y, u_z) \, du_x \, du_y \, du_z \frac{1}{S \Delta t}.$$ 

We will check the dimensions now. $(S u_x \Delta t) f(u_x, u_y, u_z) \, du_x \, du_y \, du_z$ represents a number of molecules, so it is dimensionless. Thus, the units of $p$ should be equivalent to those of $2m u_x / S \Delta t$, what remains in the expression.

$$[p] = \left[ \frac{2m u_x}{S \Delta t} \right]$$

We have

$$\frac{N}{m^2} = \frac{\text{kg} \cdot \text{m} \cdot \text{s}^{-1}}{\text{m}^2} = \frac{1}{\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}}$$

Therefore, the relation is dimensionally correct.

**Part (b)**

Bring the constants out in front of the integral and cancel $S \Delta t$.

$$p = 2m \frac{S \Delta t}{S \Delta t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} u_x^2 f(u_x, u_y, u_z) \, du_x \, du_y \, du_z$$

Substitute Eq. 1C.1-1 in for $f(u_x, u_y, u_z)$.

$$p = 2m \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} u_x^2 n \left( \frac{m}{2\pi K T} \right)^{3/2} \text{exp} \left( -\frac{m u_x^2}{2K T} \right) \, du_x \, du_y \, du_z$$

www.stemjock.com
Bring the constants in front.

\[ p = 2mn \left( \frac{m}{2\pi kT} \right)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} u_x^2 \exp \left( -\frac{mu_x^2}{2kT} \right) du_x du_y du_z \]

Let \( a = m/2kT \) so that the integral is easier to write.

\[ p = 2mn \left( \frac{a}{\pi} \right)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} u_x^2 e^{-au_x^2} du_x du_y du_z \]

Substitute \( u^2 = u_x^2 + u_y^2 + u_z^2 \).

\[ p = 2mn \left( \frac{a}{\pi} \right)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} u_x^2 e^{-a(u_x^2 + u_y^2 + u_z^2)} du_x du_y du_z \]

Split up the triple integral into three single integrals.

\[ p = 2mn \left( \frac{a}{\pi} \right)^{3/2} \left( \int_{0}^{\infty} u_x^2 e^{-au_x^2} du_x \right) \left( \int_{-\infty}^{\infty} e^{-au_y^2} du_y \right) \left( \int_{-\infty}^{\infty} e^{-au_z^2} du_z \right) \]

The last two integrals are known. Integrate the first one by parts as follows.

\[ v = u_x, \quad dw = u_x e^{-au_x^2} du \]

\[ dv = du_x, \quad w = -\frac{e^{-au_x^2}}{2a} \]

The formula being used is \( \int v dw = vw - \int w dv \), so we have

\[ p = 2mn \left( \frac{a}{\pi} \right)^{3/2} \left[ - \frac{u_x e^{-au_x^2}}{2a} \bigg|_0^{\infty} - \int_{0}^{\infty} \left( -\frac{e^{-au_x^2}}{2a} \right) du_x \right] \left( \frac{\sqrt{\pi}}{a} \right) \left( \frac{\sqrt{\pi}}{a} \right). \]

The first term in square brackets is zero. Bring the constant in front of the remaining integral.

\[ p = 2mn \left( \frac{a}{\pi} \right)^{3/2} \left( \frac{1}{2a} \int_{0}^{\infty} e^{-au_x^2} du_x \right) \left( \frac{\sqrt{\pi}}{a} \right) \left( \frac{\sqrt{\pi}}{a} \right) \]

This last integral is known.

\[ p = 2mn \left( \frac{a}{\pi} \right)^{3/2} \left( \frac{1}{2a} \cdot \frac{1}{2} \sqrt{\frac{\pi}{a}} \right) \left( \frac{\sqrt{\pi}}{a} \right) \left( \frac{\sqrt{\pi}}{a} \right) \]

Simplify the result.

\[ p = \frac{mn}{2a} \]

Substitute the expression for \( a \) and cancel terms.

\[ p = \frac{\rho n \cdot 2kT}{2\rho f} \]

Therefore,

\[ p = n kT. \]