Problem 1D.2

Force on a surface of arbitrary orientation.[5] (Fig. 1D.2) Consider the material within an element of volume $OABC$ that is in a state of equilibrium, so that the sum of the forces acting on the triangular faces $\triangle OBC$, $\triangle OCA$, $\triangle OAB$, and $\triangle ABC$ must be zero. Let the area of $\triangle ABC$ be dS, and the force per unit area acting from the minus to the plus side of dS be the vector $\boldsymbol{\pi}_n$. Show that $\boldsymbol{\pi}_n = [\mathbf{n} \cdot \boldsymbol{\pi}]$.

(a) Show that the area of $\triangle OBC$ is the same as the area of the projection $\triangle ABC$ on the yz-plane; this is $(\mathbf{n} \cdot \boldsymbol{\delta}_x)dS$. Write similar expressions for the areas of $\triangle OCA$ and $\triangle OAB$.

(b) Show that according to Table 1.2-1 the force per unit area on $\triangle OBC$ is $\boldsymbol{\delta}_x \pi_{xx} + \boldsymbol{\delta}_y \pi_{xy} + \boldsymbol{\delta}_z \pi_{xz}$. Write similar force expressions for $\triangle OCA$ and $\triangle OAB$.

(c) Show that the force balance for the volume element $OABC$ gives

$$\boldsymbol{\pi}_n = \sum_i \sum_j (\mathbf{n} \cdot \boldsymbol{\delta}_i)(\boldsymbol{\delta}_j \pi_{ij}) = [\mathbf{n} \cdot \sum_i \sum_j \boldsymbol{\delta}_i \boldsymbol{\delta}_j \pi_{ij}] \tag{1D.2-1}$$

in which the indices i, j take on the values x, y, z. The double sum in the last expression is the stress tensor $\boldsymbol{\pi}$ written as a sum of products of unit dyads and components.

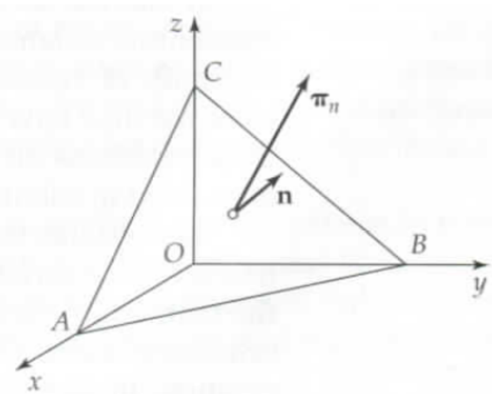

Fig. 1D.2 Element of volume $OABC$ over which a force balance is made. The vector $\boldsymbol{\pi}_n = [\mathbf{n} \cdot \boldsymbol{\pi}]$ is the force per unit area exerted by the minus material (material inside $OABC$) on the plus material (material outside $OABC$). The vector $\mathbf{n}$ is the outwardly directed unit normal vector on face ABC.

[5] M. Abraham and R. Becker, *The Classical Theory of Electricity and Magnetism*, Blackie and Sons, London (1952), pp. 44–45.